Intersection Properties of Balls in Banach Spaces and Related Topics

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Thesis submitted to the Indian Statistical Institute
in partial fulfilment of the requirements
for the award of the degree of
Doctor of Philosophy
Kolkata
2003

Acknowledgement

The work on this thesis was done under the supervision of Dr. Pradipta Bandyopadhyay of Indian Statistical Institute, Kolkata. I am grateful to him for his guidance, encouragement and active participation. This thesis materialized from our joint works.

I am also indebted to Professor T. S. S. R. K. Rao of Indian Statistical Institute, Bangalore. I had several chances of meeting him and I enjoyed a lot in working with him.

I thank Professor B. L. Lin of University of Iowa and Dr. S. Basu of Howard University, Washington D. C., for permitting me to include parts of our joint work in this thesis.

Professor A. K. Roy introduced me to the vastness of the subject 'Geometry of Banach spaces'. He also read a previous draft of thesis very carefully and had made some valuable suggestions. I take this opportunity to thank him. I also thank to all my teachers in Stat-Math who had given me several courses.

Finally, I thank the entire Stat-Math unit for providing me with a wonderful environment to work in. I enjoyed a lot in here.

Kolkata
December 2003

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Chapter 1

Introduction

1.1 Introduction

In the first part of this chapter, we explain in general terms the background and the main theme of this thesis and provide a chapter-wise summary of its principal results. In the second part, we introduce some notations and preliminaries that will be used in the subsequent chapters.

As a prototype of the properties we will study in this thesis, let us call a closed linear subspace Y of a Banach space X a (P)-subspace of X if Y has a certain property P as a subspace of X. If a Banach space X, in its canonical embedding, is a (P)-subspace of its bidual X^{**}, that often endows X with a rich geometric structure. M-embedded spaces—spaces that are M-ideals in their biduals—is a case in point. See [37, Chapter 3] for properties of such spaces. Other examples of such properties include Hahn-Banach smooth spaces, 1-complemented subspaces of the bidual etc.

On the other hand, many geometric properties of a Banach space X, in some equivalent formulation, identifies X as a (P)-subspace of X^{**}. It is often a more interesting exercise to study the general property of being a (P)-subspace of X itself. This very often yields a better understanding of the original geometric property too. We take this approach in studying some geometric properties considered in the literature. As one would expect, in doing so, we often need an algebraic, in contrast to the topological approach of the original treatments and have to develop proper tools for this analysis.

Another common feature of the properties we propose to study is that the property of being a (P)-subspace is formulated in terms of closed balls in X with centres in the subspace Y, often as an intersection property of families of such balls. In the "X in X^{**}" set-up, such properties have been studied quite extensively. For example, we may mention the work of Lindenstrauss on $\mathcal{P}_1$-spaces and L_1-predual spaces in [48], the works of Godefroy [32], Godefroy and Kalton [33], Godefroy and Saphar [34] and many other authors in the study of norm 1 complementability of X in X^{**}, existence of unique isometric predual of a dual Banach space, the ball topology and its applications, etc.

With this approach, we start with the study of the finite infinite intersection property ($IP_{f,\infty}$) studied by Godefroy in [32] and by Godefroy and Kalton in [33]. From an equivalent formulation of the $IP_{f,\infty}$, we isolate the corresponding subspace property as follows :

Definition 1.1.1. A closed linear subspace Y of a Banach space X is said to be an almost constrained (AC) subspace of X if any family of closed balls centred at points of Y that intersects in X also intersects in Y.

Clearly, a constrained (that is, 1-complemented) subspace is almost constrained. It is well known that all dual spaces and their constrained subspaces have $IP_{f,\infty}$. One of the main results in [32] is that if a Banach space X does not contain an isomorphic copy of ℓ_1 and has the $IP_{f,\infty}$, then X is a dual space and it is complemented in X^{**} by a unique norm 1 projection, which in turn ensures that X has a unique isometric predual. It should be mentioned here that the property $IP_{f,\infty}$ was first studied under the name 'finite intersection property' by Lindenstrauss in [48], where he asked whether $IP_{f,\infty}$ for a Banach space X implies X is constrained in X^{**}. This question still remains open. In [32, 35], some sufficient conditions for this are obtained.

In Chapter 2, we study AC-subspaces of Banach spaces. We give an example to show that an AC-subspace need not, in general, be constrained. We develop some tools and techniques to obtain sufficient conditions for an AC-subspace to be constrained, much in the spirit of [32, 33]. Our condition is in terms of functionals with "locally unique" Hahn-Banach (i.e., norm preserving) extensions, which improves significantly upon all existing conditions of similar type, as in [12, 35], and has a much simpler proof. As in [32, 33], these conditions actually imply the existence of a *unique* norm 1 projection. The content of this chapter is entirely taken from [7].

As noted in [5], there is a dichotomy between spaces with $IP_{f,\infty}$ and nicely smooth spaces introduced by Godefroy in [31]—if a space has both the properties then it must be reflexive. The definition of nicely smooth spaces in [31] lends itself naturally to the subspace formulation.

Definition 1.1.2. A subspace Y of a Banach space X is said to be a very non-constrained (VN) subspace of X, if for all $x \in X$,

$$\bigcap_{y \in Y} B_X[y, \|x - y\|] = \{x\},$$

where $B_X[y, \|x - y\|]$ denotes the closed ball in X with centre $y \in Y$ and radius $\|x - y\|$.

A Banach space X is said to be nicely smooth if X is a VN-subspace of X^{**}. It is obvious from the definitions that a proper subspace cannot simultaneously be very non-constrained and almost constrained. This explains the terminology. In [34], Godefroy and Saphar characterized nicely smooth spaces as follows :

Theorem 1.1.3. *[34, Lemma 2.4] For a Banach space X, the following are equivalent :*

(*a*) *X is nicely smooth.*

(*b*) *For all $x^{**} \in X^{**} \setminus X$,*

$$\bigcap_{x \in X} B_X[x, \|x^{**} - x\|] = \emptyset$$

(*c*) *X^* contains no proper norming subspace.*

In both [31] and [34], nicely smooth spaces were studied in the context of operator spaces and it was shown that a nicely smooth space has the unique extension property.

In [5], several geometric conditions ensuring nice smoothness are given : the Mazur Intersection Property (MIP) (that is, every closed bounded convex set in X is the intersection of closed balls containing it), or the Ball Generated Property (BGP) (that is, every closed bounded convex set in X is the intersection of finite union of closed balls) etc. In fact, for a separable Banach space, BGP turns out to be equivalent to nicely smooth and implies that the space is Asplund. However, nice smoothness is not inherited by subspaces. If X is nicely smooth under all equivalent renorming then it is reflexive. Further it was shown that, while nice smoothness is not a 'three space property', existence of nicely smooth renorming is.

In Chapter 3, most of the results of which are from [6], we identify some necessary and/or sufficient conditions for a subspace to be a VN-subspace. The proof of Theorem 1.1.3 $(b) \Rightarrow (a)$ in [34] depends heavily on the properties of "u.s.c. hull" of $x^{**} \in X^{**}$ considered as a function on the unit ball X_1^* of the dual X^* of X with its weak* topology. One of the key results in this chapter yields a purely algebraic proof of this result. In course of proving Theorem 1.1.3 in this general set-up, we also obtain an extension of results from [5]. For this, we again use characterization of functionals with "locally unique" Hahn-Banach extensions from Chapter 2. And this brings back some topological flavour.

In [46], the authors introduced unique ideal property which, from the point of view of application, is a natural generalization of unique extension property. We show that a VN-subspace has unique ideal property. We also note some interesting consequences of the existence of a separable VN-subspace.

If A is a subspace of $C_{\mathbb{C}}(K)$, the space of *complex* valued continuous functions on a compact Hausdorff space K, which separates points of K, we show that a necessary condition for A to be a VN-subspace of $C_{\mathbb{C}}(K)$ is that its Choquet boundary ∂A is dense in K. If A contains the constants, this condition is also sufficient.

Coming to stability results, we prove that for a family $\{X_\alpha\}$ of Banach spaces and their subspaces $\{Y_\alpha\}$, the ℓ_p $(1 \leq p \leq \infty)$ or c_0 sum of Y_α's are VN-subspaces of the sum of X_α's if and only if the same is true of each coordinate. These are natural extensions of corresponding results in [5]. We also show that for a compact Hausdorff space K, the space $C(K, Y)$, of continuous functions from K to Y equipped with the sup norm, is a VN-subspace of $C(K, X)$ if and only if Y is a VN-subspace of X. Under an assumption slightly stronger than X being nicely smooth, we show that $C(K, X)$ is a VN-subspace of $WC(K, X)$, the space of all continuous $f : K \to (X, \text{weak})$. We also show that some variants of this condition is sufficient for the space $\mathcal{K}(X, Y)$, of all compact operators from X to Y, to be a VN-subspace of the space $\mathcal{L}(X, Y)$ of all bounded operators from X to Y.

For a hyperplane H in a Banach space X, there exists a complete dichotomy between VN and AC-subspaces, namely, a hyperplane is always either a VN-subspace or an AC-subspace of X and in the later case, it is constrained. In the last section of Chapter 3, we discuss VN-hyperplanes in detail and characterize VN-hyperplanes of some classical Banach spaces.

The materials of Chapter 4 are related to the works of Godefroy and Kalton in [33] and that of Chen and Lin in [17]. For a Banach space X, the ball topology b_X is the weakest topology for which the norm closed balls are b_X-closed. Many of the concepts we have studied can also be formulated in term of the ball topology. For example, Godefroy and Kalton [33] showed that the $IP_{f,\infty}$ is equivalent to X_1 being b_X-compact. A set is ball-generated, i.e., is the intersection of finite union of closed balls, if and only if it is b_X-closed. In [33], Godefroy and Kalton studied the ball topology and its applications. Apart from studying the unique isometric predual problem, one of the major reasons for introducing the ball topology in this paper was to determine conditions on X so that for any Hausdorff linear topology τ on X for which X_1 is τ-compact, the restriction of τ to

X_1 agrees with that of a locally convex linear topology on X. They also showed that X has BGP if and only if every linear functional in X^* is ball continuous on X_1. In [17], Chen and Lin obtained characterization of ball continuous linear functionals on X_1 and answered the three question left open in [33].

In Chapter 4, we study the BGP in the subspace situation. In particular, we take a closed subspace Y of a Banach space X, and define an analogue of the ball topology by restricting the centres of balls to Y. Unlike the ball topology, this suffers from the drawback that it is not translation-invariant and thus continuity of a linear functional at 0 in this topology does not ensure that it is actually continuous everywhere. However, in the spirit of [17], we provide characterization of linear functionals which are continuous at 0 in this topology. It was proved on [5] that BGP implies nicely smooth. Analogously, we show that if every $x^* \in X^*$ is continuous at 0 in the topology we define, then Y is indeed a VN-subspace of X.

In Chapter 5, we study weighted Chebyshev centres and their relationship with intersection properties of balls. Let X be a Banach space. Let $\{a_1, a_2, \ldots, a_n\} \subseteq X$. Let $f : \mathbb{R}^n_+ \to \mathbb{R}$. Minimizers of the function $\phi : X \to \mathbb{R}$ defined by

$$\phi(x) = f(\|x - a_1\|, \|x - a_2\|, \ldots, \|x - a_n\|),$$

are called f-centres of $\{a_1, a_2, \ldots, a_n\}$. Veselý in [63] has studied Banach spaces that admits weighted Chebyshev centres for finite sets. We define weighted Chebyshev centres in a more general set-up.

Definition 1.1.4. Let X be a Banach space. For $A \subseteq X$ and $\rho : A \longrightarrow \mathbb{R}_+$, define

$$\phi_{A,\rho}(x) = \sup\{\rho(a)\|x - a\| : a \in A\}, \quad x \in X.$$

A point $x_0 \in X$ is called a weighted Chebyshev centre of A in X for the weight ρ if $\phi_{A,\rho}$ attains its minimum over X at x_0.

When A is finite, Veselý [63] has shown that that the infimum of $\phi_{A,\rho}$

over X and X^{**} are the same; if X is a dual space, A admits weighted Chebyshev centres in X for any weight ρ; and

Theorem 1.1.5. [63, Theorem 2.7] *For a Banach space X and $a_1, a_2, \ldots, a_n \in X$, the following are equivalent :*

(a) *If $r_1, r_2, \ldots, r_n > 0$ and $\bigcap_{i=1}^{n} B_{X^{**}}[a_i, r_i] \neq \emptyset$, then $\bigcap_{i=1}^{n} B_X[a_i, r_i] \neq \emptyset$.*

(b) *$\{a_1, a_2, \ldots, a_n\}$ admits weighted Chebyshev centres for all weights $r_1, r_2, \ldots, r_n > 0$.*

(c) *$\{a_1, a_2, \ldots, a_n\}$ admits f-centres for every continuous monotone coercive $f : \mathbb{R}^n_+ \to \mathbb{R}$.*

Here 'monotone' means monotone in the coordinate-wise ordering of $\mathbb{R}^n$. A careful examination of the proof shows that the 'coercive' (i.e., 'vanishing at infinity') assumption there can be dropped.

Definition 1.1.6. [63, Definition 2.8] A Banach space X is said to belong to the class (GC), denoted $X \in (GC)$, if for every $n \geq 1$ and $a_1, a_2, \ldots, a_n \in X$, the three equivalent conditions of Theorem 1.1.5 are satisfied.

Subsequently, Bandyopadhyay and Rao isolated the subspace condition from condition (a) of the above theorem to define a central subspace (C-subspace) of a Banach space in [10]. A subspace Y of X is a C-subspace if every finite family of closed balls with centres in Y that intersects in X, also intersects in Y. In particular, $X \in (GC)$ if and only if it is a C-subspace of X^{**}. Results they obtained include : (1) X is an L_1-predual if and only if whenever X is a subspace of a dual space, it is a C-subspace there; (2) if X is an L_1-predual then any compact set $A \subseteq X$ is centrable, that is, $\operatorname{diam}(A) = 2 \inf_{x \in X} \sup_{a \in A} \|x - a\|$; (3) if Y is a finite dimensional C-subspace of X, then for any extremally disconnected compact Hausdorff space K, $C(K, Y)$ is a C-subspace of $C(K, X)$.

Here, we investigate how far these results are true for a more general family of sets and thereby explore when each such set admit weighted Chebyshev centres. Extending the notion of central subspaces, we define an $\mathcal{A}$-C-subspace Y of a Banach space X with the centres of the balls coming from a given family $\mathcal{A}$ of subsets of Y, the typical examples being those of finite, compact, bounded or arbitrary sets. The first gives us the central subspace *a la* [10] and the last one gives us the AC-subspace. We extend and improve upon some results of [10, 63] in this general set-up.

We define a partial order on X and relate it to the notion of minimal points considered in [12, 35]. We also address the question of uniqueness of weighted Chebyshev centres in strictly convex spaces.

In [12], Beauzamy and Maurey considered minimal points for compact sets in a Banach space, and showed that if X is a strictly convex reflexive space, then for $A \subseteq X$ compact, the set of minimal points of A is weakly compact. We improve upon this by showing, with a much simpler proof, that the same conclusion holds if X^{**} is strictly convex and X admits weighted Chebyshev centres for all compact sets.

As illustrated in [10], L_1-predual spaces play an important role in approximation problems related to Chebyshev centres. Here we extend the characterization of L_1-predual spaces given in [10] and also identify the subspaces of a L_1-predual space which are themselves L_1-preduals. It follows from our results that in particular if an L_1-predual has $IP_{f,\infty}$ then it is a $\mathcal{P}_1$-space. By applying the celebrated Michael's Selection Theorem, we show that X is a real L_1-predual if and only if for every paracompact space T, $C(T, X)$ is a real L_1-predual. We also consider stability results as in [10]. Materials covered in this chapter have appeared recently in [8].

As a related theme, in Chapter 6, we study Mazur-like intersection properties and their relation with farthest points. Let F be a norming subspace of X^* and let σ denote the $\sigma(X, F)$ topology on X. We call a σ-closed, bounded convex set $K \subseteq X$ remotely σ-generated if K is the σ-closed convex hull of its farthest points. Lau [43, Theorem 3.3] had shown that a reflexive Banach space X has the MIP if and only if every closed bounded convex set in X is remotely generated. Under certain mild restrictions, we characterize when every member of a so-called "compatible" family of σ-closed bounded convex sets is remotely σ-generated. In particular, we show that if X is strictly convex, then every (weakly) compact convex set in X is remotely generated if and only if every such set is the intersection of closed balls containing it; and if X has the Radon-Nikodým Property (RNP), then similar result holds for w*-compact convex sets in X^*. We would like to note here that this is the only characterization available so far for weakly compact convex sets to be intersection of balls.

We also study the farthest distance map r_K of a closed bounded set K and its subdifferential ∂r_K. In [26], Fitzpatrick discussed the Gâteaux and Fréchet differentiability of r_K. In this well-known work, he showed, *inter alia*, that if both X and X^* are Fréchet smooth, then r_K is generically Fréchet differentiable. We improve upon his result and we also strengthen Westphal and Schwartz's [66] results on the range of ∂r_K. We introduce the notion of strongly farthest points and characterize strictly (respectively, locally uniformly) convex spaces as those for which every farthest point of a compact (respectively, closed bounded) convex set is a strongly farthest point. Our methods allow us to relate the differentiability of r_K to that of the norm. We also obtain a version of Preiss' Theorem [52] for ∂r_K. This part of this chapter is adapted from [9].

As further applications of the tools developed in this chapter, we study the Clarke's subdifferential $\partial^0 d_K$ (see [19]) of the distance function d_K of a closed set K and derive a *proximal normal formula.*

Definition 1.1.7. For a nonempty closed set $K \subseteq X$, the *normal cone* at $x \in X$ is defined as :

$$N_K(x) = \overline{\bigcup_{\lambda \geq 0} \lambda \partial^0 d_K(x)}^{w^*},$$

the w*-closed convex cone generated by $\partial^0 d_K(x)$.

In $\mathbb{R}^n$, $\partial^0 d_K(x)$ at any boundary point x of K has a geometrical formulation as the convex hull of the origin and the cluster points of $v_n/\|v_n\|$, where $v_n \perp K$ at boundary points x_n as $x_n \to x$ and $\|v_n\| \to 0$. The corresponding expression for the normal cone $N_K(x)$ in terms of approximating normals to K in $\mathbb{R}^n$ is called the *proximal normal formula.*

Borwein and Giles in [14] proved the *proximal normal formula* in Banach spaces for the following two cases : for an almost proximinal set in a Banach space with uniformly Gâteaux smooth norm; and, for nonempty closed sets in a smooth reflexive Banach space with a Kadec norm. Here we show that one can derive the *proximal normal formula* for almost proximinal sets in a smooth and locally uniformly convex Banach space. Our technique also allows us to prove the generic Fréchet smoothness of d_K when the norm is Fréchet smooth, strengthening the results of [14], and we derive a necessary and sufficient condition for the convexity of Chebyshev sets in a Banach space X with X and X^* locally uniformly convex. This portion of Chapter 6 is taken in parts from [25].

1.2 Notation and preliminaries

In this section, we introduce some notations and recall some definitions and results that will be used throughout this thesis. In this, we closely follow [20, 22, 37].

Let X be a Banach space. Unless otherwise mentioned, we work with real scalars. We denote by X^* the dual of X. We will identify any element $x \in X$ with its canonical image in X^{**}. By a subspace we always mean a norm closed linear subspace.

We denote by $B_X[x, r]$ the closed ball of radius $r > 0$ with centre at $x \in X$. We will simply write $B[x, r]$ if there is no confusion about the ambient space. $B_X[0, 1]$ will be denoted by X_1.

We say $A \subseteq X_1^*$ is a norming set for X if for all $x \in X$

$$\|x\| = \sup\{x^*(x) : x^* \in A\}.$$

A subspace $F \subseteq X^*$ is a norming subspace if F_1 is a norming set for X.

For a subspace Y of a Banach space X and $y^* \in Y^*$, let $\mathrm{HB}(y^*)$ be the set of all Hahn-Banach (i.e., norm preserving) extensions of y^* to X. Note that, by Hahn-Banach Theorem, $\mathrm{HB}(y^*)$ is always nonempty. A linear operator

$\Phi : Y^* \to X^*$ is called a Hahn-Banach extension operator if $\Phi(y^*) \in \mathrm{HB}(y^*)$ for all $y^* \in Y^*$.

Definition 1.2.1. A subspace Y of a Banach space X is said to be a U-subspace of X if for every $y^* \in Y^*$, the set $\mathrm{HB}(y^*)$ is singleton.

X is said to be Hahn-Banach smooth if X is a U-subspace of X^{**}.

U-subspaces were introduced and systematically studied in [50], where they were called "subspaces with Property U". Extending this, we define

Definition 1.2.2. A subspace Y of a Banach space X is said to be a weakly U-subspace of X if for every norm attaining functional $y^* \in Y^*$, $\mathrm{HB}(y^*)$ is singleton.

X is said to be weakly Hahn-Banach smooth if X is a weakly U-subspace of X^{**}.

Definition 1.2.3. For a closed set K in a Banach space X, the distance function d_K is defined as $d_K(x) = \inf\{\|z - x\| : z \in K\}$, $x \in X$. For $x \in X$, we denote the set of points of K nearest from x by $P_K(x)$, *i.e.*, $P_K(x) = \{z \in K : \|z - x\| = d_K(x)\}$. $P_K(x)$ is called the metric projection of x onto K. K is called proximinal (resp. Chebyshev) if for every $x \in X \setminus K$, $P_K(x)$ is nonempty (resp. singleton).

Definition 1.2.4. A subspace Y of a Banach space X is said to be constrained (or 1-complemented) in X if there exists a projection P of norm one on X with range Y.

A projection P on X is called an L-projection (resp. M-projection) if $\|x\| = \|Px\| + \|x - Px\|$ (resp. $\|x\| = \max\{\|Px\|, \|x - Px\|\}$), for all $x \in X$. A subspace $Y \subseteq X$ is an L- (M-) summand in X if it is the range of an L- (M-) projection in X.

A subspace Y of a Banach space X is called an M-ideal in X if $Y^\perp$ is an L-summand in X^*, where

$$Y^\perp = \{x^* \in X^* : x^*(y) = 0 \text{ for all } y \in Y\}.$$

Y is said to be a proper M-ideal if it is an M-ideal but not an M-summand.

If X is a proper M-ideal (resp. L-summand) in X^{**} then we say X is M-embedded (resp. L-embedded).

The book [37] is a standard reference for the theory of M-ideals and M-embedded spaces.

By [37, Proposition I.1.12], an M-ideal is a U-subspace.

Definition 1.2.5. Let Y be a subspace of a Banach space X. We call Y an unconditional ideal or simply an ideal in X if there exists a projection P of norm one on X^* with $\ker P = Y^\perp$.

Such a projection will be called an ideal projection.

We recall the following lemma from [45].

Lemma 1.2.6. *Let Y be a subspace of a Banach space X. The following statements are equivalent :*

(*a*) *Y is an ideal in X.*

(*b*) *Y is locally 1-complemented in X, i.e. for every finite dimensional subspace G of X and for all $\varepsilon > 0$, there is an operator $U\colon G \to Y$ such that $\|U\| \leq 1 + \varepsilon$ and $Ux = x$ for all $x \in G \cap Y$.*

(*c*) *There exists a Hahn-Banach extension operator $\Phi : Y^* \to X^*$.*

Thus, if Y is an ideal in X, one can identify Y^* with $\Phi(Y^*) \subseteq X^*$, where Φ is a Hahn-Banach extension operator.

We sketch the proof of $(a) \Leftrightarrow (c)$ above. Let $i : Y \to X$ be the inclusion map. Given a Hahn-Banach extension operator Φ, $P = \Phi \circ i^*$ is clearly an ideal projection. On the other hand, given an ideal projection P, since $\ker P = Y^\perp$, for an $y^* \in Y^*$, P takes the same value on any point of $i^{*-1}(y^*)$. Thus, $\Phi = P \circ i^{*-1}$ is well-defined and can be shown to be a Hahn-Banach extension operator.

Y is said to have unique ideal property in X if there exists at most one ideal projection P on X^*. It follows that $Y \subseteq X$ has unique ideal property if and only if there exists at most one Hahn-Banach extension operator $\Phi : Y^* \to X^*$. Hence, every U-subspace has the unique ideal property.

Note that X is always an ideal in X^{**}. The equivalent statement that X is locally 1-complemented in X^{**} is usually referred to as the Principle of Local Reflexivity (PLR). The following lemma is an easy consequence of this. We isolate it for future reference.

Lemma 1.2.7. *Let Y be an ideal in a Banach space X. Let $y_1, y_2, \dots, y_n \in Y$ and $r_1, r_2, \dots, r_n > 0$. Then*

$$\cap_{i=1}^{n} B_X[y_i, r_i] \neq \emptyset \iff \cap_{i=1}^{n} B_Y[y_i, r_i + \varepsilon] \neq \emptyset \text{ for all } \varepsilon > 0$$

In particular, if X is a Banach space, $x_1, x_2, \dots, x_n \in X$ and $r_1, r_2, \dots, r_n > 0$, then

$$\cap_{i=1}^{n} B_{X^{**}}[x_i, r_i] \neq \emptyset \iff \cap_{i=1}^{n} B_X[x_i, r_i + \varepsilon] \neq \emptyset \text{ for all } \varepsilon > 0$$

Definition 1.2.8. [42] (a) A Banach space X such that X^* is isometrically isomorphic to $L^1(\mu)$ for some positive measure μ is called an L^1-predual.

(b) A Banach space is a $\mathcal{P}_1$-space if it is constrained in every superspace.

We collect some results on L^1-predual spaces and $\mathcal{P}_1$-spaces for future reference.

Definition 1.2.9. [38] A family $\{B_X[x_i, r_i]\}$ of closed balls is said to have the weak intersection property if for all $x^* \in X_1^*$ the family $\{B_{\mathbb{R}}[x^*(x_i), r_i]\}$ has nonempty intersection in $\mathbb{R}$.

Proposition 1.2.10. *Let X be a Banach space. Then the following are equivalent :*

(a) *X is an L^1-predual space.*

(b) *[48, Theorem 6.1] and [42, Chapter 6] $X^{**}$is a $\mathcal{P}_1$-space. Also X^{**} is canonically isometric to $C(K)$, for some extremally disconnected compact Hausdorff space K and in the canonical identification extreme points of X_1^* are contained in K (extreme points correspond to indicator functions of normalized μ-atoms).*

(c) *[42, Lemma 21.3 and Theorem 21.6] For every $r > 0$ and for any finite collection of pairwise intersecting balls $\{B[x_i, r]\}$ and $\varepsilon > 0$, we have $\cap B[x_i, r + \varepsilon] \neq \emptyset$.*

(d) *[44, Proposition 4.4] If $\{B[x_i, r_i]\}$ is a family of balls with the weak intersection property such that the set of centres $\{x_i\}$ is relatively norm-compact, then $\cap B[x_i, r_i] \neq \emptyset$.*

(e) *[42, Theorem 21.6] Every finite family of pairwise intersecting balls in X intersects.*

(f) *[57, Proposition 1] X is an ideal in every superspace.*

We now recall some definitions and results related to smoothness and convexity.

Definition 1.2.11. Let f be a real-valued function defined on a Banach space X. We say f is Gâteaux differentiable at $x \in X$ if there exists an $x^* \in X^*$ such that for each $h \in X$,

$$x^*(h) = \lim_{t \to 0} \frac{f(x + th) - f(x)}{t}.$$

The functional x^* is called the Gâteaux derivative of f at x and is often denoted by $df(x)$.

If the above limit is uniform over all unit vectors $h \in X$, we say that f is Fréchet differentiable at x.

We say the norm $\| \cdot \|$ on X is Gâteaux (resp. Fréchet) differentiable if $\| \cdot \|$ is Gâteaux (resp. Fréchet) differentiable at all $x \neq 0$.

For a real-valued function f on X we define the subdifferential of f at an $x \in X$ as

$$\partial f(x) = \{x^* \in X^* : x^*(y - x) \leq f(y) - f(x), \text{ for all } y \in X\}$$

As a simple consequence of Hahn-Banach theorem, we have for each continuous convex function f on X, $\partial f(x)$ is a nonempty w*-compact set in X^*. We will call the subdifferential of the norm the duality map and use the notation $\mathcal{D}_X$ for it. Note that $\mathcal{D}_X(x) = \{x^* \in X_1^* : x^*(x) = \|x\|\}$. We will simply write $\mathcal{D}$ if there is no confusion about the ambient space.

It follows that a convex function f on X is Gâteaux differentiable at an $x \in X$ if and only if $\partial f(x)$ is a singleton. It is Fréchet differentiable at $x \in X$ if and only if it $\partial f(x)$ is a singleton and norm-norm continuous at x.

Definition 1.2.12. (a) A Banach space X is called strictly convex if $\|x\| = \|y\| = 1$ and $\|x + y\| = 2$ implies $x = y$.

(b) X is said to be locally uniformly rotund or locally uniformly convex (LUR) at $x \in X$ if $\lim \|x_n - x\| = 0$ whenever $\{x_n\} \subseteq X$ is such that $\lim \|x_n\| = \|x\|$ and $\lim \|x_n + x\| = 2\|x\|$. If the norm is LUR at each point of X, we call X LUR.

There are well-known duality relations between smoothness and convexity. If X^* is strictly convex (resp. LUR) then X is Gâteaux (resp. Fréchet) differentiable. Similar results hold for smoothness on X^*. For a comprehensive discussion of these and related results see [20, Chapter 2].

We now recall some stronger notions of extreme points, which are well known in the literature. We take our definition from [15].

Let K be a closed bounded set in X. We denote by $\partial_e K$ the set of extreme points of K. For $f \in X^*$ and $\delta > 0$, a slice of K determined by f and δ is $S(K, f, \delta) = \{k \in K : f(k) > \sup_K f - \delta\}$. A point $k_0 \in K$ is called a denting point of K if k_0 is contained in slices of K of arbitrarily small norm diameter. k_0 is called an exposed point of K if there exists $f \in X^*$ such that for all $k \in K \setminus \{k_0\}$, $f(k) < f(k_0)$ and we say f exposes K at k_0. Finally, k_0 is called an strongly exposed point if it is exposed by some $f \in X^*$ and the slices of K determined by f form a neighborhood base for the norm topology at k_0. $x \in K$ is called a point of continuity (PC) if it is a point of continuity of the identity map $Id : (K, \text{weak}) \to (K, \text{norm})$, equivalently, x is contained in a weak neighborhood of arbitrary small norm diameter.

For a w*-compact set K in X^*, if the functionals in the above definitions comes from X rather than from X^{**}, we call them w*-slices, w*-denting points, w*-exposed points and w*-strongly exposed points respectively. One also defines a w*-PC and w*-weak PC of K analogously.

A closed bounded set K in X is said to have the Radon-Nikodým Property (RNP) if every closed bounded subset of K has a denting point. X is said to have RNP if every closed bounded set in X has RNP.

Recall that a Banach space is said to be Asplund if every continuous convex function on X is Fréchet differentiable on a dense G_δ of X. The following theorem is well known.

Theorem 1.2.13. *Let X be a Banach space. Then the following are equivalent :*

(*a*) *X is an Asplund space.*

(*b*) *X^* has RNP.*

(*c*) *Every separable subspace of X has separable dual.*

We conclude this chapter by recalling some standard, and some not-so-standard, results on subspaces of $C(K)$, the space of all continuous *real or complex* valued functions on a compact Hausdorff space K. We follow the terminology from [51].

By $\mathbf{1} \in C(K)$ we mean the constant function which takes the value 1 on K. By δ_x, $x \in K$, we will denote the Dirac measure, that is, the point mass at x. Recall that

$$\partial_e C(K)_1^* = \{t\delta_x : t \in \mathbb{T},\, x \in K\},$$

where $\mathbb{T}$ denotes the unit circle in $\mathbb{C}$.

Let A be a subspace of $C(K)$ which separates points of K. The restriction of δ_x on A is denoted by ϕ_x and is called the evaluation map : $\phi_x(f) = f(x)$, $f \in A$. If A contains the constants, the state space of A is defined as

$$S_A = \{L \in A^* : \|L\| = 1 = L(\mathbf{1})\}$$

This is a weak*-compact convex subset of A_1^*. The evaluation map $\phi : K \to A_1^*$ is a homeomorphism of K into S_A. The Choquet boundary of A is defined to be the set :

$$\partial A = \phi^{-1}(\partial_e S_A)$$

For every $x \in \partial A$, δ_x is the unique Hahn-Banach extension of ϕ_x to $C(K)$. The set ∂A is a boundary for A in the sense that every $f \in A$ attains its maximum absolute value in at least one point of ∂A.

Now if A separates points of K but does not contain the constants, we cannot define the state space S_A of A. To define the Choquet boundary in this case, we observe that if $L \in \partial_e A_1^*$, then L has a Hahn-Banach extension that is in $\partial_e C(K)_1^*$. Therefore, there exists $x \in K$ and $t \in \mathbb{T}$ (not necessarily unique) such that $L = t\phi_x$. Then $\phi_x \in \partial_e A_1^*$, and

Definition 1.2.14. The Choquet boundary of A is defined to be the set :

$$\partial A = \{x \in K : \phi_x \in \partial_e A_1^*\}.$$

When A contains the constants, this coincides with the usual definition.

We will use the following lemma.

Lemma 1.2.15. *[29, Lemma 5.6] Suppose A be a subspace of $C(K)$ which separates points of K and codimension of A is n. Then $K \setminus \partial A$ contains at most n points.*

Chapter 2

Almost constrained subspaces of Banach spaces

In this chapter, we study almost constrained (AC) subspaces (see Definition 1.1.1) of Banach spaces. We give an example to show that AC-subspaces need not, in general, be constrained. We obtain some sufficient conditions for an AC-subspace to be constrained.

2.1 Basic algebraic tools

In this section, we develop some basic algebraic tools which will be used in this chapter as well as in the next. Keeping potential applications in mind, the first few results are proved for both *real* and *complex* scalars.

Definition 2.1.1. Let Y be a subspace of a normed linear space X. For $x \in X$ and $y^* \in Y^*$, put

$$\begin{aligned} U(x, y^*) &= \inf\{\operatorname{Re} y^*(y) + \|x - y\| : y \in Y\} \\ L(x, y^*) &= \sup\{\operatorname{Re} y^*(y) - \|x - y\| : y \in Y\} \end{aligned}$$

where $\operatorname{Re} y^*$ denotes the real part of y^*.

For $x^* \in X^*$, we will write $U(x, x^*)$ for $U(x, x^*|_Y)$.

Remark 2.1.2. By [31, Lemma 1], $U(x, \cdot)$ and $L(x, \cdot)$ are analogue of the "upper semicontinuous hull" and "lower semicontinuous hull" of $x \in X$ considered as a "functional" on Y_1^*. Observe that, in general, we cannot even consider $x \in X$ as a functional on Y^* as the latter may not be identifiable as a subspace of X^*.

We will use the following lemma which is an easy consequence of the proof of Hahn-Banach Theorem (see e.g., [61, section 48]).

Lemma 2.1.3. *Let Y be a subspace of a normed linear space X. Suppose $x_0 \notin Y$ and $y^* \in Y^*$, $\|y^*\| = 1$. Then $L(x_0, y^*) \leq U(x_0, y^*)$ and α lies between these two numbers if and only if there exists a Hahn-Banach extension x^* of y^* with $\operatorname{Re} x^*(x_0) = \alpha$.*

Remark 2.1.4. It is clear that for any $x^* \in X_1^*$ and $x \in X$, $L(x, x^*) \leq \operatorname{Re} x^*(x) \leq U(x, x^*)$ and an $y^* \in Y^*$, $\|y^*\| = 1$ has an unique Hahn-Banach extension to X if and only if $L(x, y^*) = U(x, y^*)$ for all $x \in X$.

Lemma 2.1.5. *Let Y be a subspace of a Banach space X. For $x_1, x_2 \in X$, the following are equivalent :*

(*a*) $x_2 \in \bigcap_{y \in Y} B_X[y, \|x_1 - y\|]$.

(*b*) *For all $x^* \in X_1^*$, $U(x_2, x^*) \leq U(x_1, x^*)$.*

Proof. Clearly, $x_2 \in \cap_{y \in Y} B_X[y, \|x_1 - y\|]$ if and only if $\|x_2 - y\| \leq \|x_1 - y\|$ for all $y \in Y$.

(*a*) $\Rightarrow$ (*b*). If for all $y \in Y$, $\|x_2 - y\| \leq \|x_1 - y\|$, then for all $x^* \in X_1^*$, $\operatorname{Re} x^*(y) + \|x_2 - y\| \leq \operatorname{Re} x^*(y) + \|x_1 - y\|$. And therefore, $U(x_2, x^*) \leq U(x_1, x^*)$.

(*b*) $\Rightarrow$ (*a*). Suppose $\|x_2 - y_0\| > \|x_1 - y_0\|$ for some $y_0 \in Y$. Then there exists $\varepsilon > 0$ such that $\|x_2 - y_0\| - \varepsilon \geq \|x_1 - y_0\|$. Choose $x^* \in X_1^*$ such that $\|x_1 - y_0\| \leq \|x_2 - y_0\| - \varepsilon < \operatorname{Re} x^*(x_2 - y_0) - \varepsilon/2$. Thus $U(x_1, x^*) \leq \operatorname{Re} x^*(y_0) + \|x_1 - y_0\| < \operatorname{Re} x^*(x_2) - \varepsilon/2 < U(x_2, x^*)$. ☐

Remark 2.1.6. Instead of X_1^*, it suffices to consider any norming set for X.

The next lemma will be needed in Chapter 3.

Lemma 2.1.7. *Let Y be a subspace of a Banach space X. For $x_1, x_2 \in X$, and $x^* \in X_1^*$, $U(x_1, x^*) - U(x_2, x^*) \leq U(x_1 - x_2, x^*)$.*

Proof. For any $x^* \in X_1^*$ and $y_1, y_2 \in Y$,

$$\begin{aligned} U(x_1, x^*) &\leq \operatorname{Re} x^*(y_1 + y_2) + \|x_1 - y_1 - y_2\| \\ &= \operatorname{Re} x^*(y_1) + \operatorname{Re} x^*(y_2) + \|(x_2 - y_2) + (x_1 - x_2 - y_1)\| \\ &\leq \operatorname{Re} x^*(y_2) + \|x_2 - y_2\| + \operatorname{Re} x^*(y_1) + \|x_1 - x_2 - y_1\| \end{aligned}$$

Since $y_1, y_2 \in Y$ are arbitrary, the result follows. □

For the rest of this chapter, we return to *real* scalars.

Definition 2.1.8. For a Banach space X and $x \in X$, let $C(x) = \{x^* \in X_1^* : U(x, x^*) = L(x, x^*)\}$ and let $C = \cap_{x \in X} C(x)$.

Proposition 2.1.9. *Let Y be a subspace of a Banach space X, $x^* \in X_1^*$ and $x_0 \in X \setminus Y$. The following are equivalent :*

(*a*) $x^* \in C(x_0)$.

(*b*) $\|x^*|_Y\| = 1$ *and every* $x_1^* \in HB(x^*|_Y)$ *takes the same value at* x_0.

(*c*) $\|x^*|_Y\| = 1$ *and if* $\{x_\alpha^*\} \subseteq X_1^*$ *is a net such that* $x_\alpha^*|_Y \to x^*|_Y$ *in the* w^**-topology of* Y^**, then* $\lim_\alpha x_\alpha^*(x_0) = x^*(x_0)$.

(*d*) $\|x^*|_Y\| = 1$ *and if* $\{x_n^*\} \subseteq X_1^*$ *is a sequence such that* $x_n^*|_Y \to x^*|_Y$ *in the* w^**-topology of* Y^**, then* $\lim x_n^*(x_0) = x^*(x_0)$.

Proof. (*a*) ⇔ (*b*). Let $\|x^*|_Y\| = \alpha$. Then $\alpha \leq \|x^*\| \leq 1$ and it suffices to show that $\alpha = 1$. Let $x_1^* \in \mathrm{HB}(x^*|_Y)$. Then $\|x_1^*\| = \alpha$ and therefore, for any $y \in Y$, $|x_1^*(x_0 - y)| \leq \alpha\|x_0 - y\| \leq \|x_0 - y\|$. It follows that

$$\begin{aligned} L(x_0, x^*) &\leq \sup\{x^*(y) - \alpha\|x_0 - y\| : y \in Y\} \leq x_1^*(x_0) \\ &\leq \inf\{x^*(y) + \alpha\|x_0 - y\| : y \in Y\} \leq U(x_0, x^*) \end{aligned}$$

Since $x^* \in C(x_0)$, equality holds everywhere.

Now if $\alpha < 1$, let $0 < \delta < d(x_0, Y)$ and $0 < \varepsilon < (1-\alpha)\delta$, then for all $y \in Y$, $(1-\alpha)\|x_0 - y\| > \varepsilon$. And therefore, for all $y \in Y$,

$$y^*(y) - \|x_0 - y\| + \varepsilon < y^*(y) - \alpha\|x_0 - y\|$$

Thus, the first inequality must be strict. Contradiction!

The result now follows from Lemma 2.1.3.

$(b) \Rightarrow (c)$. Let $\{x_\alpha^*\} \subseteq X_1^*$ be a net such that $\lim_\alpha x_\alpha^*(y) = x^*(y)$ for all $y \in Y$. It follows that any w*-cluster point of $\{x_\alpha^*\}$ is in $\mathrm{HB}(x^*|_Y)$. By (b), therefore, $\lim x_\alpha^*(x_0) = x^*(x_0)$.

$(c) \Rightarrow (d)$ is clear.

$(d) \Rightarrow (b)$. If $x_1^* \in \mathrm{HB}(x^*|_Y)$ with $x^*(x_0) \neq x_1^*(x_0)$, then the constant sequence $x_n^* = x_1^*$ clearly satisfies $\lim_n x_n^*(y) = x^*(y)$ for all $y \in Y$, but $\{x_n^*(x_0)\}$ cannot converge to $x^*(x_0)$. $\square$

Proposition 2.1.10. *Let Y be a subspace of a Banach space X. For $x^* \in X_1^*$, the following are equivalent :*

(a) $x^* \in C$.

(b) $\|x^*|_Y\| = 1$ *and* $HB(x^*|_Y) = \{x^*\}$.

(c) $\|x^*|_Y\| = 1$ *and if* $\{x_\alpha^*\} \subseteq X_1^*$ *is a net such that* $x_\alpha^*|_Y \to x^*|_Y$ *in the w*-topology of* Y^*, *then* $x_\alpha^* \to x^*$ *in the w*-topology of* X^*.

(d) $\|x^*|_Y\| = 1$ *and if* $\{x_n^*\} \subseteq X_1^*$ *such that* $x_n^*|_Y \to x^*|_Y$ *in the w*-topology of* Y^*, *then* $x_n^* \to x^*$ *in the w*-topology of* X^*.

Remark 2.1.11. Thus $C(x)$ denotes the functionals $x^* \in X_1^*$ whose restriction $x^*|_Y$ to Y have locally unique Hahn-Banach extension at $x \in X$. Similarly, C denotes the functionals $x^* \in X_1^*$ which are the unique Hahn-Banach extension of their restrictions $x^*|_Y$ to Y.

2.2 Some characterizations and a counterexample

We will use the following notation :

Notation. Let Y be a subspace of a Banach space X. For all $x \in X$,

$$\mathfrak{P}(x) = \bigcap_{y \in Y} B_Y[y, \|x - y\|].$$

Clearly, $\mathfrak{P}(y) = \{y\}$ for all $y \in Y$. And Y is an AC-subspace of X if and only if $\mathfrak{P}(x) \neq \emptyset$ for all $x \in X$. Also note that,

(a) for all $\lambda \in \mathbb{R}$, $\mathfrak{P}(\lambda x) = \lambda \mathfrak{P}(x)$.

(b) for all $y \in Y$, $\mathfrak{P}(x + y) = \mathfrak{P}(x) + y$.

Analogous to $\mathcal{O}(X)$ in [32], we now introduce the ortho-complement of Y in X.

Definition 2.2.1. For a subspace Y of a Banach space X, we define the ortho-complement $O(Y, X)$ of Y in X as

$$O(Y, X) = \{x \in X : \|x - y\| \geq \|y\| \text{ for all } y \in Y\}.$$

$O(X, X^{**})$ is denoted by $O(X)$. Note that $y \in O(Y, X)$ for $y \in Y$ if and only if $y = 0$.

Remark 2.2.2. Recall that (see e.g., [39]) for $x, y \in X$, one says x is orthogonal to y in the sense of Birkhoff (written $x \perp_B y$) if $\|x + \lambda y\| \geq \|x\|$, for all $\lambda \in \mathbb{R}$. Thus, $O(Y, X)$ is the set of all $x \in X$ such that $Y \perp_B x$. This justifies the terminology. We could have formulated most of the results in this chapter and the next, in terms of Birkhoff orthogonality also. But we did not do so, as this does not give us any better insight into the phenomenon.

The following proposition characterizes AC-subspaces.

Proposition 2.2.3. *For a subspace Y of a Banach space X, the following are equivalent :*

(a) *Y is an AC-subspace of X.*

(b) *For all $x \in X$, there exists $y \in Y$ and $z \in O(Y, X)$ such that $x = y + z$.*

(*c*) *For every subspace Z such that $Y \subseteq Z \subseteq X$ and dim$(Z/Y) = 1$, Y is constrained in Z.*

Proof. $(a) \Rightarrow (b)$. Let $x_0 \in X$. By (a), there exists $y_0 \in \mathfrak{P}(x_0)$. This implies $\|y_0 - y\| \leq \|x_0 - y\|$ for all $y \in Y$. Or, putting $u = y_0 - y$, $\|u\| \leq \|x_0 - y_0 + u\|$ for all $u \in Y$. That is, $z_0 = x_0 - y_0 \in O(Y, X)$ and $x_0 = y_0 + z_0$.

$(b) \Rightarrow (c)$. Let Z be as in (c). Then one can write $Z = \overline{\text{span}}[Y \cup \{x_0\}]$ for some $x_0 \in X$. By (b), there exists $y_0 \in Y$ and $z_0 \in O(Y, X)$ such that $x_0 = y_0 + z_0$. It follows that $Z = Y \oplus \mathbb{R}z_0$. But then, by definition of $O(Y, X)$, $\alpha z_0 + y \mapsto y$ is a norm 1 projection from Z onto Y.

$(c) \Rightarrow (a)$. By (c), for every $x \in X$, there is a norm 1 projection P_x from $Z_x = \overline{\text{span}}[Y \cup \{x\}]$ onto Y. Clearly, $P_x(x) \in \mathfrak{P}(x)$. □

Corollary 2.2.4. *A subspace Y of a Banach space X is an AC-subspace if and only if there exists a (not necessarily linear) onto map $P : X \to Y$ such that*

(*a*) $P^2 = P$

(*b*) $P(\lambda x) = \lambda P(x)$ *for all* $x \in X$, $\lambda \in \mathbb{R}$

(*c*) $P(x + y) = P(x) + y$ *for all* $x \in X$, $y \in Y$

(*d*) $\|P(x)\| \leq \|x\|$ *for all* $x \in X$

Proof. If P is as above, then clearly for any $x \in X$, $P(x) \in \mathfrak{P}(x)$. Thus, Y is an AC-subspace of X.

Conversely, let Y be an AC-subspace of X. For $z \in O(Y, X)$, let $Y_z = Y \oplus \mathbb{R}z$ and P_z be a norm 1 projection from Y_z onto Y. Observe that for $z_1, z_2 \in O(Y, X)$, either $Y_{z_1} \cap Y_{z_2} = Y$ or $Y_{z_1} = Y_{z_2}$. By Proposition 2.2.3(b), $\cup_{z \in O(Y,X)} Y_z = X$. Define $P : X \to Y$ by $P(x) = P_z(x)$, if $x \in Y_z$. Then P is well-defined and satisfies all the listed properties. □

Remark 2.2.5. Proposition 2.2.3$(a) \Leftrightarrow (c)$ for the case of $IP_{f,\infty}$ was noted in [48, Theorem 5.9]. Corollary 2.2.4 for the case of $IP_{f,\infty}$ was noted in [35, Theorem 2]. In both the cases, our proof is simpler.

Let us note that in Proposition 2.2.3(b), the representation $x = y + z$ with $y \in Y$ and $z \in O(Y, X)$ need not be unique.

Example 2.2.6. We now give an example to show that an AC-subspace need not, in general, be constrained. We follow the construction in [47]. For the sake of completeness, we include the details.

Let ω denote the ordinal number of the set of natural numbers $\mathbb{N}$. Let K be the compact metric space of all ordinals $\leq \omega^2$ in the order topology. Let

$$K_m = \{\alpha : (m-1)\omega < \alpha \leq m\omega\}$$

Define h on K as follows :

$$h(\alpha) = \begin{cases} 1 & \text{if} \quad \alpha = m\omega + 2j - 1,\ m = 0, 1, \ldots,\ j = 1, 2, \ldots \\ -1 & \text{otherwise} \end{cases}$$

also for each $n \in \mathbb{N}$, define

$$f_n(\alpha) = \begin{cases} -1 & \text{if} \quad \alpha \in K_{2m},\ m = 1, 2, \ldots, n \\ 1 & \text{otherwise} \end{cases}$$

Let V be the space of all bounded real-valued functions on $K \times \mathbb{N}$ with the sup norm. Let

$$X_0 = \{v \in V : v(\alpha, n) = v(\alpha, 1) \text{ for all } \alpha \in K, n \in \mathbb{N} \text{ and } v(\alpha, 1) \in C(K)\}.$$

Clearly, $T_0 : X_0 \to C(K)$ defined by

$$(T_0 x)(\alpha) = x(\alpha, 1), \quad \alpha \in K$$

is an onto isometry. Let Z_0 be the closed subspace of V spanned by X_0 and the functions

$$\begin{aligned} z_0(\alpha, n) &= f_n(\alpha), \quad \alpha \in K,\ n \in \mathbb{N} \\ z_k(\alpha, n) &= \delta_{k,n} h(\alpha), \quad \alpha \in K,\ n \in \mathbb{N},\ k \geq 1 \end{aligned}$$

Then

(a) If P is a projection on Z_0 onto X_0, then $\|P\| \geq 5/4$.

(b) For every Y with $X_0 \subseteq Y \subseteq Z_0$ and $\dim(Y/X_0) < \infty$ and for every $\varepsilon > 0$, there is a projection of norm $\leq 1 + \varepsilon$ from Y onto X_0.

To see this, let $\|P\| = \lambda$. For $m \geq 1$, let $g_m = \chi_{K_{2m} \times \mathbb{N}}$ be the characteristic function of the set $K_{2m} \times \mathbb{N}$. Then $g_m \in X_0$ and $\|2g_m - z_0\| = 1$ for every $m \geq 1$. Hence $Pz_0(\alpha, n) \geq 2 - \lambda$ for $\alpha \in \cup K_{2m}$, $n \in \mathbb{N}$ and by continuity of Pz_0 we have

$$Pz_0(\omega^2, 1) \geq 2 - \lambda.$$

Let $g_{m,j} = \chi_{\{(m-1)\omega+j\} \times \mathbb{N}}$. Then $g_{m,j} \in X_0$ and

$$\|\sum_{i=0}^{m} z_i + 2g_{2m+1,2j}\| = 2.$$

Hence

$$P(\sum_{i=0}^{m} z_i)(\alpha, n) \leq -2 + 2\lambda \text{ for } \alpha = 2m\omega + 2j,$$

and by continuity,

$$P(\sum_{i=0}^{m} z_i)((2m+2)\omega, 1) \leq -2 + 2\lambda.$$

Also

$$\|\sum_{i=1}^{m} z_i - g_{k,2j+1}\| = 1, \quad m, j, k \geq 1$$

as above, we obtain,

$$P(\sum_{i=1}^{m} z_i)(k\omega, 1) \geq 1 - \lambda$$

combining, we get $2 - \lambda \leq Pz_0(\omega^2, 1) \leq -3 + 3\lambda$, so that, $\lambda \geq 5/4$.

To see (b), first consider $Y_k = \text{span}(X_0 \cup \{z_0, z_1, \ldots, z_k\})$ for some $k \in \mathbb{N}$. The operator $T_k z(\alpha) = z(\alpha, k+1)$ maps Y onto $C(K)$ and T_k is an norm preserving extension of T_0. Hence there is a projection of norm 1 from Y_k onto X. Now let Y be the span of X_0 and $b_1, \ldots, b_k$ for some $b_i \in Z_0$. There is an $M < \infty$ such that

$$\sum_{1}^{k} |\lambda_i| \leq M\|x + \sum_{1}^{k} \lambda_i b_i\|, \quad x \in X_0, \lambda_i \in \mathbb{R}$$

Let $a_1, a_2, \ldots, a_k \subseteq \mathrm{span}(X_0 \cup \{z_0, z_1, \ldots\})$ such that $\|a_i - b_i\| < \varepsilon/M$. By the argument above, there is a projection Q of norm 1 from $\mathrm{span}(X_0 \cup \{a_1, a_2, \ldots, a_k\})$ onto X_0. Define P from Y onto X_0 by

$$P(x + \sum \lambda_i b_i) = Q(x + \sum \lambda_i a_i)$$

Then

$$\begin{aligned} \|P(x + \sum \lambda_i b_i)\| &\leq \|(x + \sum \lambda_i a_i)\| \\ &\leq \|(x + \sum \lambda_i b_i)\| + \varepsilon \sum |\lambda_i|/M \\ &\leq (1+\varepsilon)\|(x + \sum \lambda_i b_i)\| \end{aligned}$$

Hence P is a projection of norm $\leq 1 + \varepsilon$.

Now take $1 < \beta < 5/4$. We define a new norm $|\cdot|$ on Z_0 as the Minkowski functional of $\overline{co}(\frac{1}{\beta}(Z_{01} \cup X_{01}))$. Let Z and X be the Banach spaces Z_0 and X_0 respectively with this new norm. Observe that

(a) $\beta|z| \geq \beta\|z\| \geq |z|$, $z \in Z$.

(b) $|x| = \|x\|$, $x \in X$.

(c) If $X \subseteq Y \subseteq Z$ and P is a projection of Y onto X then $|P| = 1$ if and only if $\|P\| \leq \beta$.

Taking ε such that $1 + \varepsilon < \beta$, we see that for every Y such that $X \subseteq Y \subseteq Z$ and $\dim(Y/X) < \infty$ we have a projection of norm 1 from Y onto X and thus by Proposition 2.2.3, we have X is an AC-subspace of Z.

Clearly no such projection from Z onto X exists. Thus X is not constrained in Z.

2.3 Some sufficient conditions

We now obtain some sufficient conditions for an AC-subspace to be constrained.

Proposition 2.3.1. *For a subspace Y of a Banach space X, the following are equivalent :*

(a) *Y is an AC-subspace of X and $O(Y,X)$ is a closed subspace of X.*

(b) *Y is an AC-subspace of X and $O(Y,X)$ is a linear subspace of X.*

(c) *Y is constrained in X and for all $x \in X$, $\mathfrak{P}(x)$ is a singleton.*

Moreover, in this case, Y is constrained by a unique *norm 1 projection.*

Proof. $(a) \Rightarrow (b)$ is trivial.

$(b) \Rightarrow (c)$. Since Y is a AC-subspace of X, by Proposition 2.2.3, any $x \in X$ can be written as $x = y + z$, where $y \in Y$ and $z \in O(Y,X)$. Since both Y and $O(Y,X)$ are linear subspaces and $Y \cap O(Y,X) = \{0\}$, this representation is unique and $x \mapsto y$ is a well-defined linear map. Since $z \in O(Y,X)$, this map is of norm 1. Hence Y is constrained in X. Moreover, since $y \in \mathfrak{P}(x)$, $\mathfrak{P}(x)$ is single-valued.

$(c) \Rightarrow (a)$. Let Y be constrained in X by a norm 1 projection P and for all $x \in X$, let $\mathfrak{P}(x)$ be a singleton. Clearly, Y is an AC-subspace of X and for all $x \in X$, $\mathfrak{P}(x) = \{P(x)\}$. If $x \in \ker(P)$, then $\|x - y\| \geq \|Px - Py\| = \|y\|$ for all $y \in Y$. Thus $\ker(P) \subseteq O(Y,X)$ and since for all $x \in X$, $\mathfrak{P}(x) = \{P(x)\}$, $\ker(P) \supseteq O(Y,X)$. Thus, $O(Y,X) = \ker(P)$ is a closed subspace of X. ☐

Remark 2.3.2. (a) Even in the case of $IP_{f,\infty}$, this observation is new. [32, 33] discusses more restrictive situations, namely if for a separable X, $O(X)$ is a linear subspace of X^{**} then it is w*-closed provided X is isometric to a dual space or the ball topology on X_1 is locally linear.

(b) Can (c) be replaced by "Y is constrained by a unique norm 1 projection"?

(c) It follows from the proof that

$$\cup\{\ker(P) : P \text{ is a norm 1 projection onto } Y\} \subseteq O(Y,X).$$

Are these two sets equal?

Proposition 2.3.3. *Let Y be a subspace of X. Let $x_1, x_2 \in X$ be such that $x_1 \in \bigcap_{y \in Y} B_X[y, \|x_2 - y\|]$. Then for any $x^* \in C(x_2)$, $x^*(x_1 - x_2) = 0$.*

Proof. Let $x_1, x_2 \in X$ be such that $x_1 \in \bigcap_{y \in Y} B_X[y, \|x_2 - y\|]$. Then, by Lemma 2.1.5, for all $x^* \in X_1^*$,

$$L(x_2, x^*) \leq L(x_1, x^*) \leq U(x_1, x^*) \leq U(x_2, x^*).$$

Thus for $x^* \in C(x_2)$, equality holds. By Lemma 2.1.3, the result follows. $\square$

Corollary 2.3.4. *[12, Lemma 2] Let Y be a subspace of a Banach space X. If $x_1, x_2 \in X$ are such that for all $y \in Y$, $\|x_1 - y\| \leq \|x_2 - y\|$, then for all $y \in Y$, $\|y\| = 1$, that is a smooth point of X, and $\mathcal{D}_X(y) = \{x^*\}$, we have $x^*(x_1 - x_2) = 0$.*

Proof. It suffices to note that $x^* \in C$, and therefore, $x^* \in C(x_2)$. $\square$

The above corollary was a key tool in [12] and [35].

We now present our main result in this chapter.

Theorem 2.3.5. *Let Y be a subspace of a Banach space X. Suppose*

$$\textit{for every } x_1, x_2 \in X, \quad C(x_1) \cap C(x_2) \textit{ separates points of } Y. \tag{2.1}$$

If Y is an AC-subspace of X, then Y is constrained in X. Moreover, the projection is unique and $O(Y, X)$ is a closed subspace of X.

Proof. Since Y is an AC-subspace of X, $\mathfrak{P}(x) \neq \emptyset$ for all $x \in X$. By Proposition 2.3.3, for all $x \in X$,

$$x^*(x - y) = 0 \quad \text{for any } x^* \in C(x),\ y \in \mathfrak{P}(x). \tag{2.2}$$

Now if $y_1, y_2 \in \mathfrak{P}(x)$, then for any $x^* \in C(x)$, $x^*(x - y_1) = x^*(x - y_2) = 0$. And therefore, $x^*(y_1 - y_2) = 0$. By (2.1), $y_1 = y_2$. That is, $\mathfrak{P}(x)$ is single-valued. Let $\mathfrak{P}(x) = \{P(x)\}$. Then, it is easy to see that P satisfies the conditions of Corollary 2.2.4. So, it only remains to show that P is additive.

Let $x_1, x_2 \in X$. If $x^* \in C(x_1) \cap C(x_2)$, then by Proposition 2.1.9 (d), $x^* \in C(x_1 + x_2)$ and by (2.2), $x^*(x_1 - P(x_1)) = x^*(x_2 - P(x_2)) = x^*((x_1 + x_2) - P(x_1 + x_2)) = 0$. Therefore, $x^*(P(x_1 + x_2) - P(x_1) - P(x_2)) = 0$. By (2.1), $P(x_1 + x_2) = P(x_1) + P(x_2)$.

The rest of the result follows from Proposition 2.3.1. $\square$

It is shown in [35, Theorem 2] that an AC-subspace Y is constrained in X by a unique norm 1 projection if every unit vector in Y is a smooth point of X_1. By the following proposition and corollary, our condition is weaker.

The following result parallels the classical result of Taylor-Foguel [62, 28] that X^* is strictly convex if and only if every subspace of X is an U-subspace.

Proposition 2.3.6. *Every unit vector in Y is a smooth point of X_1 if and only if every subspace of Y is a weakly U-subspace of X. In particular, X is smooth if and only if every subspace of X is a weakly U-subspace of X.*

Proof. Suppose every unit vector in Y is a smooth point of X_1. Let Z be any subspace of Y. Suppose $z_0 \in Z$, $\|z_0\| = 1$, attains its norm at $z^* \in Z_1^*$. By assumption, z_0 is a smooth point of X_1. Now, $z^* \in \mathcal{D}_Z(z_0)$ and $\mathrm{HB}(z^*) \subseteq \mathcal{D}_X(z_0)$. Since $\mathcal{D}_X(z_0)$ is a singleton, so is $\mathrm{HB}(z^*)$. Thus, Z is a weakly U-subspace of X.

Conversely, suppose there exists $y_0 \in Y, \|y_0\| = 1$, such that $\{x_1^*, x_2^*\} \subseteq \mathcal{D}_X(y_0)$ and $x_1^* \neq x_2^*$. Let $Z = \{x \in Y : x_1^*(x) = x_2^*(x)\}$. Then $y_0 \in Z$ and therefore, $\|x_1^*|_Z\| = \|x_2^*|_Z\| = 1$. Thus, with $z^* = x_1^*|_Z$ we have z^* attains its norm at $y_0 \in Z$, but $\{x_1^*, x_2^*\} \subseteq \mathrm{HB}(z^*)$. $\square$

In the following corollary, we collect some sufficient conditions for an AC-subspace to be constrained by a unique norm 1 projection.

Corollary 2.3.7. *Let Y be an AC-subspace of X. In each of the following cases, Y is constrained in X by an unique norm 1 projection.*

(*a*) *C separates points of Y.*

(*b*) *Y is a U-subspace of X.*

(*c*) *Y is a weakly U-subspace of X.*

(*d*) *Y is an M-ideal in X.*

(*e*) *Every unit vector in Y is a smooth point of X.*

Proof. (*a*) follows from the Theorem 2.3.5. Clearly, (*b*) $\Rightarrow$ (*c*) $\Rightarrow$ (*a*). Since an M-ideal is an U-subspace, (*d*) $\Rightarrow$ (*b*). By Proposition 2.3.6, (*e*) $\Rightarrow$ (*c*). $\square$

Remark 2.3.8. In [53], it is shown that an M-ideal with $IP_{f,\infty}$ is an M-summand. Now if Y is an M-ideal in X, Y^{**} is an M-summand in X^{**}, and if moreover, Y is an AC-subspace of Y^{**}, it follows that it is an AC-subspace of X. Thus, by Corollary 2.3.7 (d), Y is constrained in X and hence is an M-summand. Thus, the result of [53] is a corollary to our results.

We will prove a more general result in Theorem 5.2.6.

Example 2.3.9. As noted in [32], the space $X = L^\infty[0,1]$ gives an example of a dual space such that there are infinitely many norm 1 projections from X^{**} onto X. This produces an example of a space with $IP_{f,\infty}$ that is constrained in X^{**}, but $O(X)$ is not a closed subspace of X^{**}. This also shows that our sufficient condition, though weaker than the known ones, is still not necessary for an AC-subspace to be constrained.

2.4 $O(Y,X)$ as closed subspace of X

We conclude the chapter with some necessary and/or sufficient conditions for $O(Y,X)$ to be a closed subspace of X. First we need a characterization of $O(Y,X)$. And we do this for both real and complex scalars.

Lemma 2.4.1. *Let X be real or complex Banach space and Y be a subspace of X. For $x \in X$, the following are equivalent :*

(a) $x \in O(Y,X)$

(b) $\ker(x)|_Y \subseteq Y^*$ *is a norming subspace for Y.*

(c) $0 \in \bigcap_{y \in Y} B_Y[y, \|x-y\|]$.

(d) *For every $x^* \in X_1^*$, $L(x,x^*) \le 0 \le U(x,x^*)$.*

(e) *For every $y^* \in Y^*$, $\|y^*\| = 1$, $L(x,y^*) \le 0 \le U(x,y^*)$.*

(f) *For every $y^* \in Y^*$, $\|y^*\| = 1$, there exists $x^* \in HB(y^*)$, such that $x^*(x) = 0$.*

(g) *For every $y^* \in Y^*$, $\|y^*\| = 1$, there exists $x^* \in HB(y^*)$, such that $\mathrm{Re}\, x^*(x) = 0$.*

Further, for a w-closed subspace $F \subseteq X^*$, $F|_Y$ is a norming subspace for Y if and only if $F_\perp \subseteq O(Y,X)$, where $F_\perp = \{x \in X : f(x) = 0$ for all $f \in F\}$.*

Proof. Let $F \subseteq X^*$ be a w*-closed subspace such that $F_\perp \subseteq O(Y,X)$. Then $F = (X/F_\perp)^*$ and therefore, it suffices to show that $\|y\| = \|y + F_\perp\| = d(y, F_\perp)$.

Clearly, $\|y\| \geq d(y, F_\perp)$. And since $F_\perp \subseteq O(Y,X)$, for any $y \in Y$ and $z \in F_\perp$, $\|y + z\| \geq \|y\|$. Thus, $d(y, F_\perp) \geq \|y\|$.

Specializing to $F = \ker(x)$, we get $(a) \Rightarrow (b)$.

$(b) \Rightarrow (a)$. Since $\ker(x)|_Y$ norms Y, $\|y\| = \|y|_{\ker(x)}\| = d(y, \mathrm{span}(x))$ for all $y \in Y$. Hence $\|x - y\| \geq \inf_{\lambda \in \mathbb{C}} \|y - \lambda x\| = \|y\|$ for all $y \in Y$. Thus, $x \in O(Y,X)$.

Now suppose $F \subseteq X^*$ is a w*-closed subspace such that $F|_Y$ is a norming subspace for Y. If $x \in F_\perp$, then $F \subseteq \ker(x)$ and therefore, $x \in O(Y,X)$. That is, $F_\perp \subseteq O(Y,X)$.

$(a) \Leftrightarrow (c)$ and $(d) \Rightarrow (e)$ are immediate from definition, while $(c) \Rightarrow (d)$ follows from Lemma 2.1.5. $(e) \Leftrightarrow (g)$ follows from Lemma 2.1.3 and $(f) \Rightarrow (g)$ is obvious.

$(a) \Rightarrow (f)$. Let $Z = \mathrm{span}(Y \cup \{x\})$. Given $y^* \in Y^*$, $\|y^*\| = 1$, define z^* on Z as

$$z^*(y + \alpha x) = y^*(y)$$

Note that since $x \in O(Y,X)$, so is αx for every scalar α. Thus

$$|z^*(y + \alpha x)| = |y^*(y)| \leq \|y\| \leq \|y + \alpha x\| \text{ for all } y \in Y,\ \alpha \in \mathbb{C}$$

Hence z^* is a Hahn-Banach extension of y^* to Z and $z^*(x) = 0$. Clearly, any Hahn-Banach extension x^* of z^* to all of X works.

$(e) \Rightarrow (a)$. For every $y^* \in Y_1^*$, $0 \leq U(x, y^*)$ implies for all $y^* \in Y_1^*$ and $y \in Y$,

$$0 \leq \mathrm{Re}\, y^*(y) + \|x - y\| \implies \mathrm{Re}\, y^*(-y) \leq \|x - y\|.$$

Since this is true for all $y^* \in Y_1^*$, $\|y\| \leq \|x - y\|$ for all $y \in Y$. That is, $x \in O(Y,X)$. □

Let $\mathcal{N} = \{F : F$ is a w*-closed subspace of X^* and $F|_Y$ is a norming subspace for $Y\}$ and $N = \cap\mathcal{N}$. Similar to [32], we observe

Proposition 2.4.2. *Let Y be a subspace of a Banach space X. $O(Y, X)$ is a closed subspace of X if and only if $N|_Y$ is a norming subspace for Y. In particular, this happens if $C|_Y$ is a norming set for Y.*

Proof. By Lemma 2.4.1, $F \in \mathcal{N}$ if and only if $F_\perp \subseteq O(Y, X)$. Thus if $N|_Y$ norms Y, then $N \in \mathcal{N}$ and hence, $N_\perp \subseteq O(Y, X)$. On the other hand, if $x \in O(Y, X)$, then $\ker(x) \in \mathcal{N}$, and hence, $N \subseteq \ker(x)$. That is, $x \in N_\perp$. Therefore, $O(Y, X) = N_\perp$, and $O(Y, X)$ is a closed subspace of X.

Conversely, if $O(Y, X)$ is a closed subspace of X and $M = O(Y, X)^\perp$, then $M_\perp = O(Y, X)$ and therefore, $M \in \mathcal{N}$. Moreover, for every $F \in \mathcal{N}$, $F_\perp \subseteq O(Y, X) = M_\perp$, and hence, $M \subseteq F$. This shows $N = M$ and $N \in \mathcal{N}$.

Now, if $C|_Y$ is a norming set for Y, then as above, $C_\perp \subseteq O(Y, X)$.

Conversely let $x \in O(Y, X)$. Let $x^* \in C$. By Lemma 2.4.1, there exists $z^* \in \mathrm{HB}(x^*|_Y)$ such that $z^*(x) = 0$. Since $x^* \in C$, $\mathrm{HB}(x^*|_Y) = \{x^*\}$, and we have $x^*(x) = 0$. Thus, $C_\perp = O(Y, X)$. $\square$

Definition 2.4.3. [65] Let Y be a subspace of a Banach space X. Let

$$A(Y) = \{x^* \in X_1^* : x^*|_Y \in \partial_e Y_1^*\}$$

We say that Y is a weakly separating subspace of X if Y separates points of $A(Y)$.

Proposition 2.4.4. *In each of the following cases, $O(Y, X)$ is a closed subspace of X,* a fortiori*, if Y is an AC-subspace, then Y is constrained by a unique norm* 1 *projection.*

(*a*) *Y is an weakly separating subspace of X.*

(*b*) *Y is a subspace of $X = C(K)$ containing the constants and separating points of K.*

Proof. (*a*). We will show that $A(Y) \subseteq C$. Since $A(Y)$ is a norming set for Y, the result follows from Proposition 2.4.2.

Let $x^* \in A(Y)$. Then clearly, $\mathrm{HB}(x^*|_Y) \subseteq A(Y)$. Since Y separates $A(Y)$, $\mathrm{HB}(x^*|_Y)$ must be singleton.

(b). We show that Y is weakly separating. In the notation of Chapter 1, for any $x^* \in A(Y)$, $x^*|_Y = t\phi_k$ for some $t \in \mathbb{T}$ and $k \in \partial Y$. Since, for any such k, $\delta(k)$ is the unique Hanh-Banach extension of ϕ_k and $\mathbf{1} \in Y$, we have $x^* = t\delta_k$ and Y separates such points. $\square$

Remark 2.4.5. In [65], it is shown that for a weakly separating subspace in $C(K)$, if there is a norm 1 projection, it must be unique. Clearly, our conclusion is stronger.

Chapter 3

Very non-constrained subspaces of Banach spaces

In this chapter, we study very non-constrained (VN) subspaces of Banach spaces (see Definition 1.1.2) and recapture many properties of nicely smooth spaces from results in this broader set-up. We discuss examples to show that there is more to VN-subspaces than mere generalizations of nicely smooth spaces. We also obtain some stability results.

3.1 Characterization and related results

We begin with our main characterization theorem for a VN-subspace. This is done for both real and complex scalars.

Theorem 3.1.1. *Let X be a real or complex Banach space and Y be a subspace of X. Then, the following are equivalent :*

(*a*) *Y is a VN-subspace of X.*

(*b*) *For any $x \in X \setminus Y$,*

$$\bigcap_{y \in Y} B_Y[y, \|x - y\|] = \emptyset.$$

(*c*) *$O(Y, X) = \{0\}$.*

(*d*) *Any* $A \subseteq X_1^*$ *such that* $A|_Y$ *is a norming set for* Y*, separates points of* X*.*

(*e*) *Any subspace* $F \subseteq X^*$ *such that* $F|_Y$ *is a norming subspace for* Y*, separates points of* X*.*

(*f*) *For all nonzero* $x \in X$*, there exists* $y^* \in Y^*$*,* $\|y^*\| = 1$ *such that every* $x^* \in HB(y^*)$ *takes non-zero value at* x*.*

Proof. Clearly, $(a) \Rightarrow (b)$.

$(b) \Rightarrow (c)$. Suppose $x \in O(Y,X)$ and $x \neq 0$. Then, $x \notin Y$ and by Lemma 2.4.1, it follows that $0 \in \bigcap_{y \in Y} B_Y[y, \|x-y\|]$, a contradiction.

$(c) \Rightarrow (a)$. Suppose $x_1, x_2 \in X$ such that

$$x_2 \in \bigcap_{y \in Y} B_X[y, \|x_1 - y\|].$$

By Lemma 2.1.5, for all $x^* \in X_1^*$, $U(x_2, x^*) \leq U(x_1, x^*)$. By Lemma 2.1.7, $0 \leq U(x_1 - x_2, x^*)$. That is, $x_1 - x_2 \in O(Y,X)$, by Lemma 2.4.1. By (c), $x_1 = x_2$. Hence Y is a VN-subspace.

$(c) \Rightarrow (d)$. Let $A \subseteq X_1^*$ be such that $A|_Y$ is a norming set for Y. By Lemma 2.4.1, $A^\perp \cap X \subseteq O(Y,X)$. By (c), therefore, $A^\perp \cap X = \{0\}$. Thus, A separates points of X.

$(d) \Leftrightarrow (e)$. Since a subspace F is norming if and only if it is the closed linear span of a norming set, this is clear.

$(d) \Rightarrow (f)$. Suppose (f) does not hold. Then there exists $x \in X$, $x \neq 0$ such that for every $y^* \in Y^*$ with $\|y^*\| = 1$, there exists $x^* \in \mathrm{HB}(y^*)$ such that $x^*(x) = 0$. Let

$$A = \{x^* \in X^* : \|x^*\| = 1,\ x^*(x) = 0\}$$

Then $A|_Y$ coincides with the unit sphere of Y^* and hence, is a norming set for Y, but A clearly does not separate x from 0.

$(f) \Rightarrow (c)$. If $x \in O(Y,X)$, then by Lemma 2.4.1, for every $y^* \in Y^*$, $\|y^*\| = 1$, there exists $x^* \in \mathrm{HB}(y^*)$ such that $x^*(x) = 0$. $\square$

We now return to real scalars.

The following observations are quite useful in applications.

Proposition 3.1.2. *Let $Y \subseteq Z \subseteq X$. If Y is a VN-subspace of X, then Z is a VN-subspace of X and Y is a VN-subspace of Z. If, moreover, Z is an AC-subspace of X, then $Z = X$.*

Proof. We observe that $O(Y,Z) \subseteq O(Y,X)$ and $O(Z,X) \subseteq O(Y,X)$. This proves the first part. In the second part, observe that Z is both a VN-subspace and an AC-subspace of X. Thus, $Z = X$. $\square$

Remark 3.1.3. Compare this with [5, Theorem 2.18] where it is proved that if X is nicely smooth, $X \subseteq E \subseteq X^{**}$ and E has $IP_{f,\infty}$, then $E = X^{**}$.

Corollary 3.1.4. *X is reflexive if and only if there is a subspace $M \subseteq X^*$ which when canonically embedded in X^{***} is a VN-subspace there.*

Proof. If X is reflexive, take $M = X^*$. Conversely, if there is an $M \subseteq X^* \subseteq X^{***}$ and M is a VN-subspace of X^{***}, then since X^* is an AC-subspace of X^{***}, we have by the above result that $X^* = X^{***}$. $\square$

Example 3.1.5. Even though the property under consideration here depends on the norm, it should be emphasized that a Banach space X may contain two isometric subspaces Y and Z such that Y is a VN-subspace of X, but Z is not.

For example, consider the usual inclusion of $c_0 \subseteq c \subseteq \ell_\infty$. The inclusion of c_0 in ℓ_∞ is the canonical embedding of c_0 in $c_0^{**} = \ell_\infty$. Since c_0 is nicely smooth, in this embedding, it is a VN-subspace of ℓ_∞. By the above result, therefore, c, in its inclusion, is a VN-subspace of ℓ_∞. However, since the action of $c^* = \ell_1$ on c is given by:

$$\langle \boldsymbol{a}, \boldsymbol{x} \rangle = a_0 \lim x_n + \sum_{n=0}^{\infty} a_{n+1} x_n, \quad \boldsymbol{a} = \{a_n\}_{n=0}^{\infty} \in \ell_1,\, \boldsymbol{x} = \{x_n\}_{n=0}^{\infty} \in c,$$

it follows that $\{\boldsymbol{a} \in \ell_1 : a_0 = 0\}$ is a proper norming subspace for c. That is, in the canonical embedding of c in $c^{**} = \ell_\infty$, c is not a VN-subspace.

Thus, there are two isometric copies of c in ℓ_∞ such that only one of them is a VN-subspace. This illustrates the need for caution in applying the above proposition.

In [5, Theorem 2.13] the authors showed that if a Banach space is nicely smooth for every equivalent renorming then it is reflexive. Here is our analogue for this:

Proposition 3.1.6. *Let Y be a subspace of a Banach space X. Then Y is a VN-subspace of X in every equivalent renorming of X if and only if $Y = X$.*

Proof. The converse being trivial, suppose $Y \neq X$. Let $x \in X \setminus Y$ and let $F = \{x^* \in X^* : x^*(x) = 0\}$. Define a new norm on Y by

$$\|y\|_1 = \sup\{x^*(y) : x^* \in F_1\} \quad \text{for } y \in Y$$

Then, for every $y \in Y$, $\|y\|_1 = d(y, E)$ where $E = \mathbb{R}x$. Thus, $\|\cdot\|_1$ is a norm on Y and $F|_Y$ a norming subspace for $(Y, \|\cdot\|_1)$.

We claim that $\|\cdot\|_1$ is an equivalent norm on Y. To see this first observe that $\|\cdot\|_1 \leq \|\cdot\|$. Conversely, let $Z = X/E$. The map $T : Y \to Z$ defined by $Ty = y + E$ is a 1-1 continuous linear map.

We now show that $T(Y)$ is closed in Z. Let $y_n + E \to x_0 + E$. Since $\|y_n - x_0 + E\| \to 0$, there exist scalars λ_n such that $\|y_n - x_0 + \lambda_n x\| \to 0$. If $\{\lambda_n\}$ is unbounded, passing through a subsequence if necessary, $\|y_n/\lambda_n + x\| \to 0$ which implies that $x \in Y$, a contradiction. Thus, $\{\lambda_n\}$ is bounded, and again passing through a subsequence if necessary, $\lambda_n \to \lambda$. Then, $y_n \to x_0 - \lambda x = y_0 \in Y$. Then $x_0 + E = y_0 + E$. It now follows from the open mapping theorem that $\|\cdot\|_1$ is an equivalent norm on Y.

Now, take the B to be the closed convex hull of the unit ball of $(Y, \|\cdot\|_1)$ with X_1. Then the Minkowski functional of B gives a equivalent norm on X whose restriction to Y is again $\|\cdot\|_1$ (see [20, Lemma II.8.1] for details). And clearly, with this norm, Y is not a VN-subspace of X. $\square$

We now try to identify some necessary and some sufficient conditions for a subspace to be a VN-subspace. We use the notations from Definition 2.1.8.

Theorem 3.1.7. *Let Y be a subspace of a Banach space X. Consider the following statements :*

(a) *C separates points of X.*

(b) *Any two distinct points in X are separated by disjoint closed balls with centres in Y.*

(b_1) *For every $x \in X$, $C(x)$ separates points of X.*

(b_2) *For every $x \in X$, $x \neq 0$, there is $x^* \in C(x)$ such that $x^*(x) \neq 0$.*

(c) *Y is a VN-subspace of X.*

Then $(a) \Rightarrow (b) \Rightarrow (c)$ and $(a) \Rightarrow (b_1) \Rightarrow (b_2) \Rightarrow (c)$.

Proof. $(a) \Rightarrow (b)$. Suppose $x_1, x_2 \in X$ are distinct. Since C separates points of X, there is an $x^* \in C$ and $\alpha \in \mathbb{R}$ such that $x^*(x_1) > \alpha > x^*(x_2)$. By definition of C, we have $\sup\{x^*(y) - \|x_1 - y\| : y \in Y\} > \alpha$. Thus there exists $y_1 \in Y$ such that $x^*(y_1) - \|x_1 - y_1\| > \alpha$. Consider the ball $B[y_1, \|x_1 - y_1\|]$. Clearly $x_1 \in B[y_1, \|x_1 - y_1\|]$ and $\inf x^*(B[y_1, \|x_1 - y_1\|]) > \alpha$.

Similarly we can get a ball $B[y_2, \|x_2 - y_2\|]$ such that $x_2 \in B[y_2, \|x_2 - y_2\|]$ and $\sup(B[y_2, \|x_2 - y_2\|]) < \alpha$. Hence the proof.

$(b) \Rightarrow (c)$ and $(a) \Rightarrow (b_1) \Rightarrow (b_2)$. Clear.

$(b_2) \Rightarrow (c)$. By (b_2), for every $x \in X$, $x \neq 0$, there is a $x^* \in C(x)$ such that $x^*(x) \neq 0$. By Proposition 2.1.9, $y^* = x^*|_Y$ is of norm 1 and every $x_1^* \in \mathrm{HB}(y^*)$ takes the same value at x. The result now follows from Theorem 3.1.1(f). □

Remark 3.1.8. If $C|_Y$ is a norming set for Y, then all the conditions are clearly equivalent. Notice that $C|_Y = \{y^* \in Y^* : \|y^*\| = 1, \mathrm{HB}(y^*)$ is singleton$\}$. Thus, this condition is satisfied if Y is an U-subspace of X.

It follows from Proposition 2.1.10 that C is the analogue of w*-weak PCs if $Y = Z$ and $X = Z^{**}$. In this case, the above condition is satisfied if Z is an Asplund space. Thus we get back much of [5, Theorem 2.10].

It is shown in [31, Lemma 5 and Lemma 6] that Hahn-Banach smooth spaces are nicely smooth. We now give an elementary example to show that, in contrast, a U-subspace need not be a VN-subspace.

Example 3.1.9. Let $X = \mathbb{R}^2$ with the Euclidean norm and $Y = \{(r,0) : r \in \mathbb{R}\}$. It is easy to see that Y is a U-subspace of X. But Y is also a constrained subspace, and therefore, not a VN-subspace of X.

Example 3.1.10. By [37, Corollary I.1.3], proper M-ideals are not constrained. In [57], it is proved that if X is an M-embedded space, then it a proper M-ideal in every even dual. Also an M-ideal is an U-subspace. However, by Corollary 3.1.4, such an X cannot be a VN-subspace of $X^{(4)}$. Thus, we get another example of a U-subspace which is not a VN-subspace.

In fact, this example shows that even a proper M-ideal need not be a VN-subspace. However, an M-embedded space, being Hahn-Banach smooth, is always nicely smooth.

Let us now try to understand why such examples work.

Definition 3.1.11. A subspace Y of a Banach space X is said to be a $(*)$-subspace of X if the set

$$A = \{x^* \in X_1^* : \|x^*|_Y\| = 1\}$$

separates points of X.

Proposition 3.1.12. *Let Y be a subspace of a Banach space X.*

(*a*) *If Y is a VN-subspace, then Y is a $(*)$-subspace.*

(*b*) *If Y is a $(*)$-subspace as well as a U-subspace of X, then Y is a VN-subspace.*

Proof. By Theorem 3.1.1(d), if Y is a VN-subspace, A separates points of X. And if Y is a U-subspace, $A \subseteq C$ and therefore, if Y is $(*)$-subspace, C separates points of X. ☐

Here are some natural examples of $(*)$-subspaces.

(a) X is a $(*)$-subspace of X^{**}.

(b) If $Y \subseteq Z \subseteq X$ and Y is a $(*)$-subspace of X, then Z is a $(*)$-subspace of X and Y is a $(*)$-subspace of Z.

(c) For any two Banach spaces X and Y, $\mathcal{K}(X, Y)$ is a $(*)$-subspace of $\mathcal{L}(X, Y)$. To see this, note that $A \supseteq \{x^{**} \otimes y^* : x^{**} \in X^{**},\ y^* \in Y^*,\ \|x^{**}\| = \|y^*\| = 1\}$, which separates points in $\mathcal{L}(X, Y)$.

(d) If Y is a $(*)$-subspace of Z, then for any Banach space X, $X \otimes_\pi Y$ is a $(*)$-subspace of $X \otimes_\pi Z$. To see this, consider the set $A = \{x^* \otimes z^* : x^* \in X^*,\ \|x^*\| = 1,\ z^* \in A(Y)\}$, where $A(Y) = \{z^* \in Z_1 : \|z^*|_Y\| = 1\}$. Since Y is a $(*)$-subspace of Z, $A(Y)$ separates points of Z, and hence, A separates points in $X \otimes_\pi Z$.

In particular, $X \otimes_\pi Y$ is a $(*)$-subspace of $X \otimes_\pi Y^{**}$.

The following result from [56] produces more examples of $(*)$-subspaces.

Lemma 3.1.13. *[56, Lemma 1] Let Y be an ideal in X such that there is an ideal projection P with $P(X^*)$ is a norming subspace of X^*. Then there is an isometric embedding of X in Y^{**} whose restriction to Y is the canonical embedding of Y in Y^{**}.*

Thus we have,

(e) Any ideal Y satisfying the condition of Lemma 3.1.13 is a $(*)$-subspace. To see this, note that if Y is such an ideal, then Y^* is isometrically identified with PX^*. Since $\{x^* \in (PX^*)_1 : \|x^*\| = 1\} \subseteq A$, the set A is, in fact, norming for X.

In particular, $C(K, X)$ is a $(*)$-subspace of $WC(K, X)$.

Definition 3.1.14. A Banach space X has the unique extension property (UEP), if the only operator $T \in \mathcal{L}(X^{**})$ such that $\|T\| \leq 1$ and $T|_X = Id_X$ is $T = Id_{X^{**}}$.

As mentioned in the Introduction, it was shown in [34, Proposition 2.5] that nicely smooth spaces have the UEP. Here we show that

Proposition 3.1.15. *Let Y be a VN-subspace of X. Then*

(a) *the only operator $T \in \mathcal{L}(X)$ such that $\|T\| \leq 1$ and $T|_Y = Id_Y$ is $T = Id_X$.*

(*b*) *Y has the unique ideal property in X.*

Proof. (a). Let $x \in X$. $\|Tx - y\| = \|Tx - Ty\| \leq \|T\|\|x - y\| \leq \|x - y\|$ for all $y \in Y$. But then, as in the proof of Theorem 3.1.1, $x - Tx \in O(Y, X)$ and since $O(Y, X) = \{0\}$ we have the result.

(b). Let P_i, $i = 1, 2$ be two ideal projections for Y. It is enough to show that for all $x \in X_1$, $P_1^*(x) = P_2^*(x)$.

We make the following observations :

(i) Let $\sigma_i = \sigma(X, P_i X^*)$, $i = 1, 2$ be the topologies induced on X by $P_i X^*$. Since Y is a VN-subspace of X and $P_i X^*$ are norming for Y, we have (X_1, σ_i) are Hausdorff spaces. Also Y_1 is σ_i-dense in X_1 (see [37], Remark 1.13).

(ii) $P_i^*|_Y = Id|_Y$.

(iii) $P_1^* = P_2^* P_1^*$.

(iv) On $(P_1^* X^{**})_1$, we can consider the two topologies τ_1 and τ_2 induced by $P_1 X^*$ and $P_2 X^*$ respectively. It is easy to note that these two are compact Hausdorff topologies on $(P_1^* X^{**})_1$ and from (iii), the identity map is τ_1-τ_2 continuous. Thus these two topologies are identical.

(v) $\tau_i|_Y = \sigma_i|_Y$, $i = 1, 2$.

Now, given $x \in X_1$, take a net $\{y_\alpha\} \subseteq Y_1$ such that $y_\alpha \xrightarrow{\sigma_1} x$. Since σ_1 is Hausdorff, x is the unique σ_1-cluster point of $\{y_\alpha\}$. Therefore, $y_\alpha \xrightarrow{\sigma_2} x$ also. Thus for all $x^* \in X^*$, $(P_1^* y_\alpha)(x^*) = x^*(y_\alpha) \longrightarrow (P_1^* x)(x^*)$ and $(P_2^* y_\alpha)(x^*) = x^*(y_\alpha) \longrightarrow (P_2^* x)(x^*)$. Thus $P_1^* x = P_2^* x$ as desired. □

Remark 3.1.16. In case of X in X^{**}, as noted in [46], (a) $\Leftrightarrow$ (b). We do not know if (a) $\Rightarrow$ (b). However, (b), in general, does not imply (a). See Remark 3.4.2 below.

Example 3.1.17. Observe that a 1-dimensional subspace is always constrained, and therefore, cannot be a VN-subspace. Can a space have a finite dimensional VN-subspace? Recall that a Banach space is called polyhedral if the unit ball of its each finite dimensional subspace is a convex polytope.

It is easy to see that in a polyhedral Banach space, for example c_0, finite dimensional subspaces cannot be VN-subspaces, since their duals have only finitely many extreme points. But in c we can exhibit a two-dimensional VN-subspace.

Consider the subspace $Y \subseteq c$ spanned by $\boldsymbol{x} = (\sin\frac{1}{n})$ and $\boldsymbol{y} = (\cos\frac{1}{n})$. Taking vectors of the form $\sin 1/k \cdot \boldsymbol{x} + \cos 1/k \cdot \boldsymbol{y}$, one can see that any norming subspace for Y in ℓ_1 contains all the unit vectors e_n. Hence Y is a VN-subspace.

We now discuss some consequences of the existence of a separable VN-subspace. We will need the following result from [55, p 208].

Proposition 3.1.18. *Let K be a compact metric space and $f \in WC(K, X)$. There exists a sequence $\{f_n\} \subseteq C(K, X)$ such that $f_n \to f$ pointwise in the norm topology.*

Theorem 3.1.19. *Let X be a Banach space with a separable VN-subspace Y. Then,*

(*a*) *there is a countable set $\{\xi_n^*\}$ of norm 1 functionals which separates points of X.*

(*b*) *Weakly compact subsets of X are metrizable.*

(*c*) *Let K be a compact Hausdorff space. Then every $f \in WC(K, X)$ is Baire class-1, i.e., there exists a sequence $\{f_n\} \subseteq C(K, X)$ such that $f_n \to f$ pointwise.*

Proof. (*a*). Let $\{y_n\}$ be a dense subset of the unit sphere of Y. Let $y_n^* \in \mathcal{D}(y_n)$. Then $\{y_n^*\}$ is norming for Y. For each y_n^*, choose $x_n^* \in \mathrm{HB}(y_n^*)$. Then $\overline{\mathrm{span}}\{x_n^*\}$ is a subspace of X^* which is norming for Y.

Since Y is a VN-subspace of X, $\overline{\mathrm{span}}\{x_n^*\}$ separates points of X. Now $\overline{\mathrm{span}}\{x_n^*\}$ being norm separable, there is a norm dense set $\{\xi_n^*\} \subseteq \overline{\mathrm{span}}\{x_n^*\}$ which separates points of X.

(*b*). Let K be a weakly compact subset of X. Since $\{\xi_n^*\}$ separates points of K and are weakly continuous, we have the result.

(c). Let $W = f(K)$. W is a weakly compact subset of X. Hence by (b), it is weakly metrizable.

Now take $F : W \to X$ to be the identity map on W. By Proposition 3.1.18, there exists $F_n \in C(W, X)$ such that $F_n(w) \to w$ for each $w \in W$. Note that $w = f(k)$ for some $k \in K$. Define $f_n(k) = F_n(w)$. It is easy to see that f_n satisfies the required conditions. □

The following result is also immediate.

Proposition 3.1.20. *Suppose Y is a separable subspace of a Banach space X such that Y is a VN-subspace of X^{**}. Then X_1^{**} is w^*-metrizable.*

Remark 3.1.21. If a Banach space X satisfies the hypothesis of Proposition 3.1.20, then (X, w) is σ-fragmentable (see [41] for details).

Recall that a Banach space X is called weakly compactly generated (WCG) if there exists a weakly compact set $K \subseteq X$ such that $X = \overline{\text{span}}(K)$.

Theorem 3.1.22. *Let X be a WCG Banach space with a separable VN-subspace. Then X itself is separable.*

Proof. It is well known that any separable subspace of a WCG space is actually contained in a separable constrained subspace (see, e.g., [20, page 238]). Hence the result follows from Proposition 3.1.2. □

3.2 VN-subspaces of $C_{\mathbb{C}}(K)$

In this section, we consider a closed subspace A of $C_{\mathbb{C}}(K)$ which separates points of K and examine when A is a VN-subspace of $C_{\mathbb{C}}(K)$. Naturally, we work with complex scalars in this section.

The following is the main result in this section:

Theorem 3.2.1. *Let A be a closed subspace of $C_{\mathbb{C}}(K)$ which separates points of K. If A is VN-subspace of $C_{\mathbb{C}}(K)$ then $\overline{\partial A} = K$.*

Moreover, if A contains the constants, then the converse is also true.

Proof. Suppose $\overline{\partial A} \neq K$. We can get an nonzero $f \in C_{\mathbb{C}}(K)$ such that $f|_{\partial A} = 0$. We show $f \in O(A, C_{\mathbb{C}}(K))$.

Let $h \in A$. There exists $x \in \partial A$ such that $|h(x)| = \|h\|$. Then

$$\|f - h\| \geq |f(x) - h(x)| = |h(x)| = \|h\|.$$

Thus $f \in O(A, C_{\mathbb{C}}(K))$.

Now suppose A contains the constants. Let $\overline{\partial A} = K$. Consider $f \in O(A, C_{\mathbb{C}}(K))$. For $x \in \partial A$, $\phi(x)$ has unique Hahn-Banach extension $\delta(x)$ to $C_{\mathbb{C}}(K)$. By Lemma 2.4.1, we have $f(x) = 0$. Thus $f = 0$ and A is VN-subspace of $C_{\mathbb{C}}(K)$. This completes the proof. $\square$

Remark 3.2.2. (a) When A is a subspace of $C_{\mathbb{C}}(K)$ that does not separate points of K, we do not have an analogue of the Choquet boundary. However, in this case, we can define an equivalence relation on K as follows : for $x, y \in K$, say $x \sim y$ if $f(x) = f(y)$ for all $f \in A$. Let $\widetilde{K}$ be the quotient of K with this equivalence relation. Then one can realize A isometrically as a subspace $\widetilde{A}$ of $C(\widetilde{K})$ and $\widetilde{A}$ separates points in $\widetilde{K}$. It is not hard to see that if A is a VN-subspace of $C_{\mathbb{C}}(K)$ then $\widetilde{A}$ is a VN-subspace of $C(\widetilde{K})$, that is, by Theorem 3.2.1, $\overline{\partial \widetilde{A}} = \widetilde{K}$.

(b) Any function algebra A can always be considered as a function algebra over its Shilov boundary $\overline{\partial A}$. It is evident from the above theorem that A is always a VN subspace of $C(\overline{\partial A})$. Also in general, the class of function algebras in $C_{\mathbb{C}}(K)$ with their Shilov boundaries equal to K are quite large, and this contains, for instance, all logmodular algebras (see [16]).

We now give an example to show that if A does not contain the constants, the condition $\overline{\partial A} = K$ does not ensure that A is a VN-subspace of $C_{\mathbb{C}}(K)$.

Example 3.2.3. Let X be any *complex* Banach space. Then X naturally embeds as an point separating subspace in $C_{\mathbb{C}}(\overline{\partial_e X_1^*}^{w^*})$. Obviously we have $\partial X = \partial_e X_1^*$ and clearly $\mathbf{1} \notin X$ where $\mathbf{1}$ is the constant function in $C_{\mathbb{C}}(\overline{\partial_e X_1^*}^{w^*})$. Now for $x \in X$, get $x^* \in \partial_e X_1^*$ such that $x^*(x) = -\|x\|$. Then $\|\mathbf{1} - x\|_\infty \geq |(\mathbf{1} - x)(x^*)| = 1 + \|x\| > \|x\|$. Thus $\mathbf{1} \in O(X, C_{\mathbb{C}}(\overline{\partial_e X_1^*}^{w^*}))$ and X is not an VN-subspace of $C_{\mathbb{C}}(\overline{\partial_e X_1^*}^{w^*})$.

We now characterize M-ideals in $C_{\mathbb{C}}(K)$ which are VN-subspaces. Recall that any M-ideal in $C_{\mathbb{C}}(K)$ is of the form $M = \{f \in C_{\mathbb{C}}(K) : f|_D = 0\}$ for some closed set $D \subseteq K$ (see [37, Example 1.4 (a)]).

Proposition 3.2.4. *Let $D \subseteq K$ be a closed set. Let $M = \{f \in C_{\mathbb{C}}(K) : f|_D = 0\}$. Then M is a VN-subspace of $C_{\mathbb{C}}(K)$ if and only if $K \setminus D$ is dense in K.*

Proof. Suppose $K \setminus D$ is dense in K. Let $f \in O(M, C_{\mathbb{C}}(K))$. $\{\phi(x) : x \in X \setminus D\}$ are norm 1 functionals on M and the M-ideal M is a U-subspace of $C_{\mathbb{C}}(K)$. Thus by Lemma 2.4.1, $f(x) = 0$ for all $x \in K \setminus D$. Since $K \setminus D$ is dense in K, we have $f = 0$.

Conversely, if $K \setminus D$ is not dense in K, there is a $h \in C_{\mathbb{C}}(K)$ such that $h \neq 0$ and $h|_{K \setminus D} = 0$. For any $f \in M$, there exists $x \in K \setminus D$ such that $|f(x)| = \|f\|$. Thus, arguing as in the proof of Theorem 3.2.1, $h \in O(M, C_{\mathbb{C}}(K))$. Thus M cannot be VN-subspace of $C_{\mathbb{C}}(K)$. □

3.3 Stability results

Theorem 3.3.1. *Let Γ be an index set. For all $\alpha \in \Gamma$, let Y_α be a subspace of X_α. Then the following are equivalent :*

(a) *For all $\alpha \in \Gamma$, Y_α is a VN-subspace of X_α.*

(b) *For some $1 \leq p \leq \infty$, $\oplus_{\ell_p} Y_\alpha$ is a VN-subspace of $\oplus_{\ell_p} X_\alpha$.*

(c) *For all $1 \leq p \leq \infty$, $\oplus_{\ell_p} Y_\alpha$ is a VN-subspace of $\oplus_{\ell_p} X_\alpha$.*

(d) *$\oplus_{c_0} Y_\alpha$ is a VN-subspace of $\oplus_{\ell_\infty} X_\alpha$.*

(e) *$\oplus_{c_0} Y_\alpha$ is a VN-subspace of $\oplus_{c_0} X_\alpha$.*

Proof. $(c) \Rightarrow (b)$ is trivial.

(b) or $(e) \Rightarrow (a)$. Let $X = \oplus X_\alpha$ and $Y = \oplus Y_\alpha$, where the sum is any of c_0- or ℓ_p- $(1 \leq p \leq \infty)$ sum. It is immediate that if for every $\alpha \in \Gamma$, $x_\alpha \in O(Y_\alpha, X_\alpha)$, then $x = (x_\alpha) \in O(Y, X)$. Hence $O(Y, X) = \{0\}$ implies $O(Y_\alpha, X_\alpha) = \{0\}$ for all $\alpha \in \Gamma$.

$(a) \Rightarrow (c)$ for $1 \leq p < \infty$. Let $O(Y_\alpha, X_\alpha) = \{0\}$ for all α. Let $x \in \oplus_{\ell_p} X_\alpha$ be nonzero. Let α_0 be such that $x_{\alpha_0} \neq 0$. Since Y_{α_0} is VN-subspace of X_{α_0}, there exists $y_{\alpha_0} \in Y_{\alpha_0}$ such that $\|x_{\alpha_0} - y_{\alpha_0}\| < \|y_{\alpha_0}\|$. Get $\varepsilon > 0$ such that $\|x_{\alpha_0} - y_{\alpha_0}\|^p < \varepsilon + \|y_{\alpha_0}\|^p$. Then there exists a finite $\Gamma_0 \subseteq \{\alpha \in \Gamma : \ x_\alpha \neq 0\}$ such that $\alpha_0 \in \Gamma_0$ and $\sum_{\alpha \notin \Gamma_0} \|x_\alpha\| < \varepsilon$. If $\alpha \in \Gamma_0$ then there is $y_\alpha \in Y_\alpha$ such that $\|x_\alpha - y_\alpha\| < \|y_\alpha\|$. Define $y \in \oplus_{\ell_p} Y_\alpha$ by,

$$y_\alpha = \begin{cases} y_\alpha & \text{if} \quad \alpha \in \Gamma_0 \\ 0 & \text{otherwise} \end{cases}$$

Easy to check that $\|x - y\|_p < \|y\|_p$.

$(d) \Rightarrow (e)$ and (c) for $p = \infty$. This follows from Proposition 3.1.2.

$(a) \Rightarrow (d)$. Let $x \in \oplus_\infty X_\alpha$ be nonzero. Get α_0 such that $x_{\alpha_0} \neq 0$. As before, there exists $y_{\alpha_0} \in Y_{\alpha_0}$ such that $\|x_{\alpha_0} - y_{\alpha_0}\| < \|y_{\alpha_0}\|$. By triangle equality we have for any $\lambda \geq 1$, $\|x_{\alpha_0} - \lambda y_{\alpha_0}\| < \|\lambda y_{\alpha_0}\|$. Thus we may assume that $\|y_{\alpha_0}\| > \|x\|_\infty$. Define $y \in \oplus_0 Y_\alpha$ by,

$$y_\alpha = \begin{cases} y_{\alpha_0} & \text{if} \quad \alpha = \alpha_0 \\ 0 & \text{otherwise} \end{cases}$$

A simple calculation shows $\|x - y\|_\infty < \|y\|_\infty$. $\square$

Since $(\oplus_{\ell_p} X_\alpha)^{**} = \oplus_{\ell_p} X_\alpha^{**}$ for $1 < p < \infty$ and $(\oplus_{c_0} X_\alpha)^{**} = \oplus_{\ell_\infty} X_\alpha^{**}$, the following corollary is immediate.

Corollary 3.3.2. *Let $\{X_\alpha)$ be a family of Banach spaces. Then,*

(a) *[5, Theorem 3.1] $X = \oplus_{\ell_p} X_\alpha$ $(1 < p < \infty)$ is nicely smooth if and only if for each α, X_α is nicely smooth.*

(b) *[5, Theorem 3.3] $X = \oplus_{c_0} X_\alpha$ is nicely smooth if and only if for each α, X_α is nicely smooth.*

(c) *$\oplus_{c_0} X_\alpha$ is a VN-subspace of $\oplus_{\ell_\infty} X_\alpha$.*

We now consider $C(K, Y)$ in $C(K, X)$ where Y is a subspace of X. We need the following result.

Lemma 3.3.3. *Let K be compact Hausdorff space. Let Y be a subspace of X. Consider $y^* \otimes \delta_k \in C(K, Y)_1^*$ where $y^* \in Y^*$, $\|y^*\| = 1$ and $k \in K$. Then $HB(y^* \otimes \delta_k) = HB(y^*) \otimes \delta_k$.*

Proof. Let $G = y^* \otimes \delta_k$. Then $G \in C(K, Y)^*$ and $\|G\| = 1$. Let $F \in \mathrm{HB}(G)$, then $\|F\| = 1$ and considering the total variation of F, it is not difficult to see that F is also a point mass at k. That is, $F = x^* \otimes \delta_k$, where $x^* \in \mathrm{HB}(y^*)$. □

Theorem 3.3.4. *Let K be a compact Hausdorff space. Let Y be a subspace of X. $C(K, Y)$ is a VN-subspace of $C(K, X)$ if and only if Y is a VN-subspace of X.*

Proof. Observe that if $x \in O(Y, X)$, then the constant function $x \in O(C(K, Y),\ C(K, X))$. Hence if $C(K, Y)$ is a VN-subspace of $C(K, X)$, then Y is a VN-subspace of X.

Conversely, let Y be a VN-subspace of X. By Theorem 3.1.1, it suffices to show that for all nonzero $f \in C(K, X)$, there exists $G \in C(K, Y)^*$, $\|G\| = 1$ such that every $F \in \mathrm{HB}(G)$ takes nonzero value at f.

Let $f \neq 0 \in C(K, X)$. Choose $k_0 \in K$ such that $f(k_0) \neq 0$. Since Y is a VN-subspace of X, by Theorem 3.1.1, there exists $y^* \in Y^*$, $\|y^*\| = 1$, such that every $x^* \in \mathrm{HB}(y^*)$ takes nonzero value at $f(k_0)$. Define G by

$$G = y^* \otimes \delta_{k_0}$$

Let $F \in \mathrm{HB}(G)$. Then by the above lemma, $F = x^* \otimes \delta_{k_0}$, for some $x^* \in \mathrm{HB}(y^*)$. Hence $F(f) = x^*(f(k_0)) \neq 0$. □

Remark 3.3.5. Compare this result with the result of [5] that $C(K, X)$ is nicely smooth if and only if X is nicely smooth and K is finite.

We need the following lemma from [46]. We include the proof for completeness.

Lemma 3.3.6. *Suppose Y is a subspace of Z.*

(*a*) *Consider* $X \otimes Y$ *as a subspace of* $\mathcal{L}(X^*, Z)$. *Let* x^* *be a* w^**-denting point of* X_1^*. *Then, for any* $y^* \in Y^*$, $\|y^*\| = 1$,

$$HB(x^* \otimes y^*) = x^* \otimes HB(y^*).$$

(*b*) *Consider* $\mathcal{K}(X, Y)$ *as a subspace of* $\mathcal{L}(X, Z)$. *Let* x *be a denting point of* X_1. *Then, for any* $y^* \in Y^*$, $\|y^*\| = 1$,

$$HB(x \otimes y^*) = x \otimes HB(y^*).$$

Proof. Clearly, $x^* \otimes \mathrm{HB}(y^*) \subseteq \mathrm{HB}(x^* \otimes y^*)$. For the converse, let $\phi = x^* \otimes y^*$ and let $\psi \in \mathrm{HB}(\phi)$. We show $\psi \in \overline{\{x^*\} \otimes Z_1^*}^{w^*}$ where we take the w*-topology on $\mathcal{L}(X^*, Z)^*$.

Since x^* is a w*-denting point, given $\varepsilon > 0$, we can choose $\delta > 0$ and $x \in X$ such that $x^*(x) = 1$, $\|x\| \le 1 + \delta\varepsilon$ and

$$(\|w^*\| \le 1 \text{ and } w^*(x) > 1 - \delta) \Rightarrow \|w^* - x^*\| \le \varepsilon.$$

Fix $T \in \mathcal{L}(X^*, Z)$ with $\|T\| = 1$. Choose $y \in Y$ such that $y^*(y) = 1$ and $\|y\| \le 1 + \delta\varepsilon$. Take $S = x \otimes y \in X \otimes Y$. Then $\psi(S) = 1$. Since

$$\psi \in \mathcal{L}(X^*, Z)_1 = \overline{co}^{w^*}(X_1^* \otimes Z_1^*),$$

get $\psi_1 \in co(X_1^* \otimes Z_1^*)$ such that

$$\psi_1(S) > 1 - \delta^2\varepsilon^2 \quad \text{and} \quad |\psi_1(T) - \psi(T)| < \varepsilon$$

Write $\psi_1 = \sum_{i=1}^m \alpha_i x_i^* \otimes z_i^*$ where $x_i^* \in X_1^*$, $z_i^* \in Z_1^*$, $\alpha_i > 0$ and $\sum_{i=1}^m \alpha_i = 1$. We may suppose that $z_i^*(y) \ge 0$ for all i. Let $J = \{i : x_i^*(x) >$

$1-\delta\}$. Then we have,

$$\begin{aligned}
1-\delta^2\varepsilon^2 < \psi_1(S) &= \sum_{i=1}^{m}\alpha_i x_i^*(x)z_i^*(y)\\
&= \sum_{i\in J}\alpha_i x_i^*(x)z_i^*(y)+\sum_{i\notin J}\alpha_i x_i^*(x)z_i^*(y)\\
&\leq \sum_{i\in J}\alpha_i(1+\delta\varepsilon)^2+\sum_{i\notin J}\alpha_i(1+\delta\varepsilon)(1-\delta)\\
&= (1+\delta\varepsilon)[(1+\delta\varepsilon)\sum_{i\in J}\alpha_i+(1-\delta)\sum_{i\notin J}\alpha_i]\\
&\leq (1+\delta\varepsilon)[1+\delta\varepsilon-\delta\sum_{i\notin J}\alpha_i].
\end{aligned}$$

Thus

$$1-\delta\varepsilon < 1+\delta\varepsilon-\delta\sum_{i\notin J}\alpha_i,$$

that is

$$\sum_{i\notin J}\alpha_i < 2\varepsilon$$

Let

$$\eta=\sum_{i=1}^{m}\alpha_i x^*\otimes z_i^*,$$

then

$$\begin{aligned}
|\eta(T)-\psi(T)| &\leq |\psi_1(T)-\psi(T)|+\|\psi_1-\eta\|\\
&\leq \varepsilon+\|\sum_{i=1}^{m}\alpha_i(x^*-x_i^*)\otimes z_i^*\|\\
&\leq \varepsilon+\sum_{i\in J}\alpha_i\|x^*-x_i^*\|+\sum_{i\notin J}\alpha_i\|x^*-x_i^*\|\\
&< \varepsilon+\varepsilon+4\varepsilon=6\varepsilon
\end{aligned}$$

Thus we have

$$\psi\in\overline{\{x^*\}\otimes Z_1^*}^{w^*}.$$

Now choose a net $\{z_\alpha^*\}\subseteq Z_1^*$ such that $x^*\otimes z_\alpha^*\to\psi$ in w*-topology of $\mathcal{L}(X^*,Z)^*$. By passing to a subnet if necessary, we may assume that $z_\alpha^*\to z^*$

for some $z^* \in Z_1^*$. Let $x \in X$ be such that $x^*(x) = 1$. Then for any $y \in Y$,

$$y^*(y) = y^*(y)x^*(x) = \phi(x \otimes y) = \lim_\alpha (z_\alpha^* \otimes x^*)(x \otimes y) = z^*(y),$$

so we get that $z^*|_Y = y^*$. Thus $z^* \in \mathrm{HB}(y^*)$.

For any $T \in \mathcal{L}(X^*, Z)$ we have,

$$\psi(T) = \lim_\alpha (z_\alpha^* \otimes x^*)(T) = \lim_\alpha z_\alpha^*(Tx^*) = (x^* \otimes z^*)(T),$$

That is, $\psi = x^* \otimes z^*$. This completes the proof.

The proof of (b) can be obtained essentially along the same line. $\square$

Remark 3.3.7. Note that $\delta(k)$ is a w*-denting point of $C(K)_1^*$ if and only if k is an isolated point of K. As we have seen in Lemma 3.3.3 above, even without any such assumption, we have $\mathrm{HB}(y^* \otimes \delta(k)) = \mathrm{HB}(y^*) \otimes \delta(k)$.

Theorem 3.3.8. *Let X and Z be Banach spaces and Y is a subspace of Z. If $X \otimes_\varepsilon Y$ is a VN-subspace of $\mathcal{L}(X^*, Z)$, then Y is a VN-subspace of Z. And if w*-denting points of X_1^* separate points of X^{**}, then the converse also holds.*

In particular, if X satisfies this condition, then $C(K, X)$ is a VN-subspace of $\mathcal{L}(X^*, C(K))$, and hence, also of $WC(K, X)$.

Proof. Suppose $X \otimes_\varepsilon Y$ is a VN-subspace of $\mathcal{L}(X^*, Z)$. Then by Proposition 3.1.2, $X \otimes_\varepsilon Y$ is a VN-subspace of $X \otimes_\varepsilon Z$. We show that in that case, Y is a VN-subspace of Z.

Let $F \subseteq Z^*$ be a subspace such that $F|_Y$ norms Y. By definition of the injective norm, $X_1^* \otimes Y_1^*$ is a norming set for $X \otimes_\varepsilon Y$. It follows that $X_1^* \otimes F_1$ is a norming set for $X \otimes_\varepsilon Y$. If F does not separate points of Z, there is a $z \in Z$ such that $z^*(z) = 0$ for all $z^* \in F$. Take any $x \in X$, $x \neq 0$. Observe that for any $x^* \in X^*$, $x^* \otimes z^*(x \otimes z) = 0$. Hence $X_1^* \otimes F_1$ does not separate points of $X \otimes Z$. This contradicts the assumption that $X \otimes_\varepsilon Y$ is a VN-subspace of $X \otimes_\varepsilon Z$.

Now, suppose w*-denting points of X_1^* separates points of X^{**} and Y is a VN-subspace of Z.

As before, by Theorem 3.1.1, it suffices to show that for all nonzero $T \in \mathcal{L}(X^*, Z)$, there exists $\phi \in (X \otimes Y)^*$, $\|\phi\| = 1$ such that every $\Phi \in \mathrm{HB}(\phi)$ takes nonzero value at T.

Let $T \in \mathcal{L}(X^*, Z)$, $T \neq 0$. Passing to T^*, get a w*-denting point x^* of X_1^* such that $Tx^* \neq 0$. Then, since Y is a VN-subspace of Z, there is $y^* \in Y^*$, $\|y^*\| = 1$ such that if $z^* \in \mathrm{HB}(y^*)$, we have $z^*(Tx^*) \neq 0$. By Lemma 3.3.6, $\mathrm{HB}(x^* \otimes y^*) = x^* \otimes \mathrm{HB}(y^*)$ and therefore, $\phi = x^* \otimes y^*$ works.

Since $C(K, X) = C(K) \otimes_\varepsilon X$, it follows from the first part that $C(K, X)$ is a VN-subspace of $\mathcal{L}(X^*, C(K))$.

For the other assertion, we embed $WC(K, X)$ in $\mathcal{L}(X^*, C(K))$. For $f \in WC(K, X)$ define $T_f \in \mathcal{L}(X^*, C(K))$ by $(T_f x^*)(k) = x^*(f(k))$. Then we have $C(K, X) \subseteq WC(K, X) \subseteq \mathcal{L}(X^*, C(K))$. Hence by Proposition 3.1.2, we have the result. $\square$

Corollary 3.3.9. *If $C(K, X)$ is a VN-subspace of $\mathcal{L}(X^*, C(K))$, then X is nicely smooth. And if X is Asplund (or, separable), the converse also holds.*

Proof. If $C(K, X)$ is a VN-subspace of $\mathcal{L}(X^*, C(K))$, then $C(K, X)$ is a VN-subspace of $\mathcal{K}(X^*, C(K)) = C(K, X^{**})$. It follows from Theorem 3.3.4 that X is nicely smooth.

If X is Asplund (or, separable) as well as nicely smooth, then w*-denting points of X_1^* separate points of X^{**}, and therefore, $C(K, X)$ is a VN-subspace of $\mathcal{L}(X^*, C(K))$. $\square$

Remark 3.3.10. Recall a Banach space X has Schur Property if the weakly convergent sequences in X are norm convergent. It is known that $C(K, X) = WC(K, X)$ for any K if and only if X has the Schur property [54]. And that when K is infinite and X fails the Schur property, $C(K, X)$ is not constrained in $WC(K, X)$ [23]. Will $C(K, X)$ be a VN-subspace of $WC(K, X)$ in such case?

Some examples where $C(K, X)$ is an M-ideal in $WC(K, X)$ have been discussed in [54]. Since $C(K, X)$ is a $(*)$-subspace of $WC(K, X)$, and M-ideals are U-subspaces, by Proposition 3.1.12, it follows that any such

$C(K,X)$ is a VN-subspace of $WC(K,X)$. However, the space X in all these examples is an Asplund space and hence, the condition of Theorem 3.3.8 holds.

A Banach space X with the MIP satisfies the hypothesis of Theorem 3.3.8. By [40, Corollary 2.8], any Banach space embeds isometrically into a Banach space with the MIP. Now, since the Schur property is hereditary, starting with any Banach space Z failing the Schur property, we can produce a Banach space X with the MIP and failing the Schur property. This will produce examples when $C(K,X)$ is a proper VN-subspace of $WC(K,X)$.

Proposition 3.3.11. *Suppose denting points of X_1 separate points of X^*. Let Y be a VN-subspace of Z. Then $\mathcal{K}(X,Y)$ is a VN-subspace of $\mathcal{K}(X,Z)$.*

Proof. This follows from Lemma 3.3.6(b) along the same line as in Theorem 3.3.8. □

Finally, we consider $\mathcal{K}(X,Y)$ in $\mathcal{L}(X,Y)$. There are discussions in the literature (see e.g. [37, Chapter 6]) on the situation when $\mathcal{K}(X,Y)$ is an M-ideal in $\mathcal{L}(X,Y)$. Since $\mathcal{K}(X,Y)$ is a $(*)$-subspace, by Proposition 3.1.12 again, it follows that each such $\mathcal{K}(X,Y)$ is a VN-subspaces of $\mathcal{L}(X,Y)$. Here are some more situations when $\mathcal{K}(X,Y)$ is a VN-subspace of $\mathcal{L}(X,Y)$.

Theorem 3.3.12. *Suppose X and Y are Banach spaces. If*

(a) *w^*-denting points of Y_1^* separate points of Y^{**}, or*

(b) *denting points of X_1 separate points of X^*,*

then $\mathcal{K}(X,Y)$ is a VN-subspace of $\mathcal{L}(X,Y)$.

Proof. Let $S = A \otimes B$, where in (a), A denotes the extreme points of X_1^{**} and B denotes the set of w*-denting points of Y_1^*; and in (b), A denotes the set of denting points of X_1 and B denotes the set of extreme points of Y_1^*. By Lemma 3.3.6, in both cases, $S \subseteq C$ (Definition 2.1.8). And by the assumptions on X and Y, in both cases, S separates points of $\mathcal{L}(X,Y)$. □

Remark 3.3.13. The assumption "(w*-)denting points separate points" in the above discussion allows us to make use the special form of Hahn-

Banach extensions on tensor product spaces as in Lemma 3.3.6. It would be interesting to know if the results are true even without this assumption.

3.4 VN-hyperplanes

In this section, we discuss VN-hyperplanes in Banach spaces. As mentioned before, there is a dichotomy between VN- and AC-hyperplanes and a AC-hyperplane is constrained. Thus identifying all VN-hyperplanes amounts to identifying all constrained hyperplanes.

Proposition 3.4.1. *For a hyperplane H in a Banach space X, the following are equivalent :*

(a) *H is a VN-subspace of X.*

(b) *H is not an AC-subspace of X.*

(c) *the only operator $T \in \mathcal{L}(X)$ such that $\|T\| \leq 1$ and $T|_H = Id_H$ is $T = Id_X$.*

(d) *H is not constrained in X.*

Proof. Clearly, $(a) \Rightarrow (b) \Rightarrow (d)$ and $(a) \Rightarrow (c) \Rightarrow (d)$.

$(d) \Rightarrow (a)$. Let $x^* \in X^*$, $\|x^*\| = 1$ be such that $H = \ker x^*$. Suppose H is not a VN-subspace in X. Then there is a $x_0 \in O(H, X)$, $x_0 \neq 0$. Since $O(H, X)$ is closed under scalar multiplication, we may assume $x^*(x_0) = 1$.

Clearly, $P : X \to X$ defined by $P(x) = x - x^*(x)x_0$ is a bounded linear projection onto H. It suffices to show that $\|P(x)\| \leq \|x\|$ for all $x \in X$.

Let $x \in X$. Then $x^*(x)x_0 \in O(H, X)$ and therefore, $\|x\| = \|x^*(x)x_0 + P(x)\| \geq \|P(x)\|$. □

Remark 3.4.2. Observe that we cannot replace (c) above by "H has the unique ideal property in X".

For example, let K be a compact Hausdorff space and $X = C(K)$. Let $k_0 \in K$ be an isolated point and let $H = \{f \in C(K) : f(k_0) = 0\}$. Then H is an M-summand in X and therefore, is not a VN-subspace. However,

H is an M-ideal and hence a U-subspace of X. As noted in Chapter 1, it follows that H has the unique ideal property in X.

However, such a situation cannot occur for X in X^{**} and we obtain

Corollary 3.4.3. *Let X be a Banach space such that* $\dim(X^{**}/X) = 1$. *Then the following are equivalent :*

(a) *X is not nicely smooth.*

(b) *X has the $IP_{f,\infty}$.*

(c) *X is constrained in X^{**}.*

(d) *X fails the UEP.*

An example of a Banach space which satisfies the equivalent conditions of the above corollary is the James space J. The space J satisfies the following properties: (see [24, Chapter 6 and Chapter 9] for details).

(a) $\dim(J^{**}/J) = 1$.

(b) J is constrained in the James tree space JT which is a dual Banach space.

Thus J has $IP_{f,\infty}$ and since it a hyperplane in J^{**}, J is constrained in J^{**}.

We now characterize VN-hyperplanes in some classical Banach spaces.

Proposition 3.4.4. *For a Banach space X, the following are equivalent :*

(a) *X is a Hilbert space.*

(b) *No proper subspace of X is a VN-subspace.*

(c) *No hyperplane in X is a VN-subspace.*

Proof. In a Hilbert space, every subspace is constrained, hence no proper subspace is a VN-subspace. Thus $(a) \Rightarrow (b) \Rightarrow (c)$.

$(c) \Rightarrow (a)$. If no hyperplane is a VN-subspace, by Proposition 3.4.1, every hyperplane is constrained. It is well known (see e.g., [1, Theorem 12.7]) that this implies X is a Hilbert space. □

At the other end of the spectrum are spaces in which all hyperplanes are VN-subspaces. Examples of such spaces are available even among reflexive spaces. Let us recall the following result.

Theorem 3.4.5. *[12, Proposition VI.3.1] Let $1 < p < \infty$, $p \neq 2$ and $1/p + 1/q = 1$. Let $f \in L_q(\Omega, \Sigma, \mu)$, $f \neq 0$. Then the hyperplane $\ker f$ is constrained in $L_p(\Omega, \Sigma, \mu)$ if and only if f is of the form $f = \alpha\chi_A + \beta\chi_B$, where A and B are atoms of μ and $\alpha, \beta \in \mathbb{R}$.*

Thus for μ nonatomic, the spaces $L_p(\mu)$, $1 < p < \infty$, $p \neq 2$, provide examples of reflexive spaces in which all hyperplanes are VN-subspaces. Since there are constrained subspaces in these spaces, this also shows that intersection of VN-subspaces need not be a VN-subspace.

Even for $L_1(\mu)$ with μ nonatomic, it is known that, no subspace of finite co-dimension is constrained (see [37, Corollary IV.1.15]). In particular, no hyperplane is constrained. Thus again, all hyperplanes in $L_1(\mu)$ are VN-subspaces.

Coming to the sequence spaces, Theorem 3.4.5 also shows that for $1 < p < \infty$, $p \neq 2$ and $1/p + 1/q = 1$, for $\phi \in \ell_q$, the hyperplane $\ker\phi$ is constrained in ℓ_p if and only if at most 2 coordinates of ϕ are nonzero.

The same statement is also true for ℓ_1. This was proved by [13, Theorem 3]. But their argument is quite involved. Here is a simple proof.

Proof. Suppose $\phi = (s_1, s_2, 0, 0, \ldots) \in \ell_\infty$ and $H = \ker\phi$. Let

$$\boldsymbol{z} = \frac{1}{|s_1| + |s_2|}(sgn(s_1), sgn(s_2), 0, 0, \ldots) \in \ell_1.$$

Then $\phi(\boldsymbol{z}) = 1$ and it is not difficult to verify that the projection defined by $P(\boldsymbol{x}) = \boldsymbol{x} - \phi(\boldsymbol{x})\boldsymbol{z}$ is of norm 1.

Conversely, suppose Let $\phi = (s_1, s_2, s_3, \ldots) \in \ell_\infty$ has at least three nonzero coordinates and $H = \ker\phi$. Without loss of generality, assume s_1, s_2, s_3 are nonzero. We will show that H cannot be an AC-subspace. Since $\boldsymbol{x_0} = (1/s_1, 1/s_2, 1/s_3, 0, 0, \ldots) \notin H$, if H were an AC-subspace, we would have an $\boldsymbol{y_0} \in \cap_{y \in H} B_H[\boldsymbol{y}, \|\boldsymbol{x_0} - \boldsymbol{y}\|]$. Let $\boldsymbol{y_0} = (y_1, y_2, y_3, \ldots)$. Note that

$\boldsymbol{z_0} = \boldsymbol{x_0} - \boldsymbol{y_0} \in O(H, X)$. Now if we put $\boldsymbol{y} = (1/s_1, 1/s_2, -2/s_3, 0, 0, \ldots) - \boldsymbol{y_0}$, then $\boldsymbol{y} \in H$. And therefore, $\|\boldsymbol{z_0} - \boldsymbol{y}\| \geq \|\boldsymbol{y}\|$. That is,

$$\left|\frac{3}{s_3}\right| \geq \left|\frac{1}{s_1} - y_1\right| + \left|\frac{1}{s_2} - y_2\right| + \left|\frac{2}{s_3} + y_3\right| + \sum_{i=4}^{\infty} |y_i|.$$

And hence,

$$\left|\frac{1}{s_3} - y_3\right| \geq \left|\frac{1}{s_1} - y_1\right| + \left|\frac{1}{s_2} - y_2\right| + \sum_{i=4}^{\infty} |y_i|.$$

Similarly taking $\boldsymbol{y} = (1/s_1, -2/s_2, 1/s_3, 0, 0, \ldots) - \boldsymbol{y_0}$, we get

$$\left|\frac{1}{s_2} - y_2\right| \geq \left|\frac{1}{s_1} - y_1\right| + \left|\frac{1}{s_3} - y_3\right| + \sum_{i=4}^{\infty} |y_i|,$$

and taking $\boldsymbol{y} = (-2/s_1, 1/s_2, 1/s_3, 0, 0, \ldots) - \boldsymbol{y_0}$, we get

$$\left|\frac{1}{s_1} - y_1\right| \geq \left|\frac{1}{s_2} - y_2\right| + \left|\frac{1}{s_3} - y_3\right| + \sum_{i=4}^{\infty} |y_i|.$$

This is surely not possible. □

Coming to c_0, it is shown in [13, Theorem 1] that for $\phi = (s_1, s_2, s_3, \ldots) \in \ell_1$ with $\|\phi\| = 1$, the hyperplane $\ker \phi$ is constrained in c_0 if and only if $|s_n| \geq 1/2$ for some n. Thus, whenever $|s_n| < 1/2$ for all n, the hyperplane $\ker \phi$ is a VN-subspace.

It follows from the results of [11] that for $\phi \in \ell_\infty^*$ with $\|\phi\| = 1$, if we write $\phi = \phi_1 + \phi_2$, where $\phi_1 = (s_1, s_2, s_3, \ldots) \in \ell_1$ and $\phi_2 \in c_0^\perp$, then the hyperplane $\ker \phi$ is a VN-subspace of ℓ_∞ if and only if $|s_n| < 1/2$ for all n.

Question 3.1. *Can one characterize $x^* \in X^*$ such that $\ker x^*$ is a VN-subspace of X?*

We conclude this chapter with a discussion of VN-hyperplanes in $C_{\mathbb{C}}(K)$.

The following result is immediate from Proposition 3.2.4 and Proposition 3.1.2.

Proposition 3.4.6. *Let λ be a regular Borel measure on K such that $K \setminus$ supp(λ) is dense in K, then $\ker \lambda$ is a VN-hyperplane in $C_{\mathbb{C}}(K)$.*

Coming to the general case, let H be the kernel of $\lambda \in C_{\mathbb{C}}(K)^*_1$. Two cases are possible:

Case 1. *H does not separate points in K.*

Then there exist $x_1, x_2 \in K$ such that $f(x_1) = f(x_2)$ for all $f \in H$ and therefore, $H = ker(\delta_{x_1} - \delta_{x_2})$.

Claim : H is constrained if and only if one of x_1 or x_2 is an isolated point of K.

To prove this, note that if one of x_1 or x_2 is isolated, say x_1, then 1_{x_1}, the indicator function of $\{x_1\}$, is continuous and hence for $h \in C_{\mathbb{C}}(K)$, $h \to h - (h(x_1) - h(x_2))1_{x_1}$ is a contractive projection on $C_{\mathbb{C}}(K)$ with range H.

Conversely, if none of x_1 and x_2 is an isolated point of K, by Proposition 3.4.6, H is a VN-subspace of $C_{\mathbb{C}}(K)$. Thus we have the result by Proposition 3.4.1.

Case 2. *H separates points in K.*

In this case, we can state our result in terms of the Choquet boundary ∂H of H. Note that by Lemma 1.2.15, $K \setminus \partial H$ contains at most one point. Thus there are two possibilities here : either $K = \partial H$ or $K = \partial H \cup \{x\}$ for some $x \in K$.

In the first case, if $\mathbf{1} \in H$, then by Theorem 3.2.1, H is a VN-hyperplane in $C_{\mathbb{C}}(K)$. We do not know what happens in case $\mathbf{1} \notin H$.

In the latter case, if x is an isolated point, then by Theorem 3.2.1, H is a constrained subspace irrespective of H contains $\mathbf{1}$ or not.

If x is not isolated and $\mathbf{1} \in H$, then $\overline{\partial H} = K$ and by Theorem 3.2.1, H is a VN-hyperplane. Again we do not know the result for the case when $\mathbf{1} \notin H$.

We conclude the chapter with the following general result.

Theorem 3.4.7. *Let K be a compact Hausdorff space with no isolated point. Then there is no constrained hyperplane in $C_{\mathbb{C}}(K)$. Thus all hyperplanes are VN-subspaces.*

Proof. Let H be a hyperplane in $C_{\mathbb{C}}(K)$. If H does not separate points in

K the result follows from the previous discussion.

Now let H separates points of K. Note that, by Lemma 1.2.15, $K \setminus \partial H$ is at most singleton. Since K has no isolated point, $\overline{\partial H} = K$. We will show if K has no isolated point, then the set

$$K_0 = \{x \in \partial H : \delta_x \text{ is the unique Hahn-Banach extension of } \phi_x\}$$

is dense in K. Thus the result will follow as in the proof of Theorem 3.2.1.

Let $H = \ker \lambda$, where $\lambda \in C_{\mathbb{C}}(K)^*$, $\|\lambda\| = 1$. The following cases are possible:

Case 1. λ is purely non-atomic.

In this case, we claim that $K_0 = \partial H$.

To see this let $x \in K$, and ν be any Hahn-Banach extension of ϕ_x. Then for some $\alpha \in \mathbb{C}$,

$$\nu - \delta_x = \alpha\lambda.$$

Evaluating at x, we have,

$$\nu(x) - 1 = 0.$$

Hence, $\nu(x) = 1$, that is $\nu = \delta_x$.

Case 2. λ is purely atomic.

Let $\lambda = \sum_{i \in I} \alpha_i \delta_{x_i}$, where I is a finite or countable set. In this case, we claim that there can be at most two points in ∂H that are not in K_0.

If $x \notin \{x_i : i \in I\}$, we can argue as in Case 1 to show that $x \in K_0$

Now, assume without loss of generality, $x_1 \notin K_0$. Let ν be a Hahn-Banach extension of ϕ_{x_1} different from δ_{x_1}. Then

$$\nu - \delta_{x_1} = \alpha\lambda = \alpha \sum_{i \in I} \alpha_i \delta_{x_i}$$

for some $\alpha \neq 0$. Evaluating at x_i we have,

$$\nu(x_1) = 1 + \alpha\alpha_1 \text{ and } \nu(x_i) = \alpha\alpha_i, i \neq 1.$$

Thus we have,

$$1 = |1 + \alpha\alpha_1| + \sum_{i \in I} |\alpha\alpha_i| = |1 + \alpha\alpha_1| + |\alpha|(1 - |\alpha_1|)$$

that is,

$$|1+\alpha\alpha_1| - |\alpha\alpha_1| = 1 - |\alpha|.$$

Since $\alpha \neq 0$, we have, $|\alpha_1| \geq 1/2$.

Now if there is another point, say $x_2 \notin K_0$, a similar calculation shows $|\alpha_2| \geq 1/2$. Since $\sum |\alpha_i| = 1$, we have,

$$\lambda = \alpha_1\delta_{x_1} + \alpha_2\delta_{x_2}$$

and for any other point in $x \in K$, $x \in K_0$.

Case 3. $\lambda = \mu + \sum \alpha_i \delta_{x_i}$, where μ is purely non-atomic and $|\mu|(K) > 0$.

In this case, we claim that there can at most one point in ∂H such that $x \notin K_0$.

If $x \notin \{x_i\}$, then as before, $x \in K_0$.

If for two points from $\{x_i\}$, say x_1 and x_2, is in K_0, then arguing as in Case 2, we have both $|\alpha_1| \geq 1/2$ and $|\alpha_2| \geq 1/2$, which contradicts $|\mu|(K) > 0$. This completes the proof of the theorem. $\square$

Chapter 4

Generalization of the Ball Generated Property

4.1 Generalization of the ball topology

In this chapter, we study some generalizations of the BGP. Throughout this chapter we only consider *real* Banach spaces. Recall that a Banach space X is said to have the BGP if every closed bounded convex set in X is ball generated, that is, intersection of finite union of closed balls in X. Equivalently, X has the BGP if and only if every closed bounded convex set in X is closed in the ball topology on X. The ball topology b_X on X is the weakest topology such that the norm closed balls are closed. In an attempt to find out a proper analogue of the BGP in the subspace situation, we define :

Definition 4.1.1. Let Y be a subspace of a Banach space X. We denote by $b_{Y,X}$ the weakest topology on X such that the norm closed balls in X with centres in Y are closed.

It is well known that X has BGP if and only if all $f \in X^*$ are continuous on (X_1, b_X). The main difficulty with $b_{Y,X}$ is as soon as we restrict the centers to Y, the topology no longer remains translation-invariant and the techniques of [33] fails. In particular, a $x^* \in X^*$ which is continuous in $b_{Y,X}$ at 0, may not be so at an arbitrary $x \in X$. However, we obtain necessary and sufficient conditions for a $x^* \in X^*$ to be $b_{Y,X}$-continuous in X_1 at 0 and show that if all $x^* \in X^*$ are $b_{Y,X}$-continuous in X_1 at 0, then Y is a VN-subspace of X. This parallels the result in [32], that BGP implies nicely smooth.

Remark 4.1.2. (a) A subbase with closed sets for $b_{Y,X}$ is given by $\{B_X[y,r] : y \in Y, r > 0\}$.

(b) $b_{Y,X} \subseteq b_X$, $b_{Y,X}|_Y = b_Y$, and $b_{Y,X}$ is translation-invariant if and only if $b_{Y,X} = b_X$.

We recall the proof of the following lemma, essentially due to Phelps.

Lemma 4.1.3. *For a normed linear space X, if for unit vectors $f, g \in X^*$ and $\varepsilon > 0$, $\{x \in X_1 : f(x) > \varepsilon\} \subseteq \{x \in X : g(x) > 0\}$, then $\|f - g\| < 2\varepsilon$.*

Proof. By the hypothesis, if $x \in X_1$ and $g(x) = 0$, then $|f(x)| \leq \varepsilon$. That is, $\|f|_{\ker(g)}\| \leq \varepsilon$. By Hahn-Banach Theorem, there exists $h \in X^*$ such that $\|h\| \leq \varepsilon$ and $h \equiv f$ on $\ker(g)$. It follows that $f - h = tg$ for some $t \in \mathbb{R}$. Then $\|f - tg\| = \|h\| \leq \varepsilon$. Now, if $y \in \{x \in X_1 : f(x) > \varepsilon\}$, then $g(y) > 0$ and

$$\varepsilon < f(y) \leq (f - tg)(y) + tg(y) \leq \|f - tg\| + tg(y) \leq \varepsilon + tg(y)$$

It follows that $t > 0$. And

$$|1 - t| \leq |\|f\| - \|tg\|| \leq \|f - tg\| \leq \varepsilon$$

Now,

$$\|f - g\| \leq \|f - tg\| + |1 - t| \leq \varepsilon + \varepsilon = 2\varepsilon$$

□

The following theorem extends [18, Theorem 2.1] with much simpler proof, which we adopt from [3].

Theorem 4.1.4. *Let $A \subseteq X$ be a bounded subset of X. Then there is a closed ball B in X with centre in Y such that $A \subseteq B$ and $0 \notin B$ if and only if $d(0, A) > 0$ and there exists a w*-slice of X_1^* determined by some element $y \in Y$ which is contained in*

$$\{f \in X_1^* : f(x) > 0 \textit{ for all } x \in A\}.$$

Proof. Let $A \subseteq B_X[y_0, r]$ and $0 \notin B_X[y_0, r]$. Then $\|y_0\| > r$. Clearly, $d(0, A) \geq \|y_0\| - r > 0$.

Let $S = \{f \in X_1^* : f(y_0) > r\}$. Then S is a w*-slice of X_1^* determined by $y_0 \in Y$. And if $g \in S$, then for any $x \in A$,

$$g(y_0 - x) \leq \|y_0 - x\| \leq r$$

and hence,

$$g(x) \geq g(y_0) - r > 0$$

Thus,

$$S \subseteq \bigcap_{x \in A} \{f \in X_1^* : f(x) > 0\}.$$

Conversely, let $\delta = d(0, A) > 0$ and let $y_0 \in Y$, $\|y_0\| = 1$ and $0 < \varepsilon < 1$ be such that

$$\{f \in X_1^* : f(y_0) > \varepsilon\} \subseteq \bigcap_{x \in A} \{f \in X_1^* : f(x) > 0\}.$$

Let $M = \sup\{\|x\| : x \in A\}$. By the proof of Lemma 4.1.3, for $x \in A$, there exists $t \in \mathbb{R}$ such that $1 - \varepsilon \leq t \leq 1 + \varepsilon$ and

$$\left\| \frac{tx}{\|x\|} - y_0 \right\| \leq \varepsilon.$$

Then for $n \geq M/(1 - \varepsilon)$,

$$\begin{aligned}
\|x - ny_0\| &\leq \left\| x - \frac{\|x\|}{t} y_0 \right\| + \left| \frac{\|x\|}{t} - n \right| \leq \frac{\varepsilon \|x\|}{t} + n - \frac{\|x\|}{t} \\
&= n - \frac{\|x\|}{t}(1 - \varepsilon) \leq n - \frac{\delta(1 - \varepsilon)}{1 + \varepsilon}
\end{aligned}$$

Therefore,

$$A \subseteq B_X\left[ny_0, n - \frac{\delta(1-\varepsilon)}{1+\varepsilon}\right] \text{ and clearly, } 0 \notin B_X\left[ny_0, n - \frac{\delta(1-\varepsilon)}{1+\varepsilon}\right].$$

□

Definition 4.1.5. Let $f \in X^*$. We say f is $b_{Y,X}$-continuous at 0 if for $\{x_\alpha\} \subseteq X_1$ and $x_\alpha \to 0$ in $b_{Y,X}$ we have $f(x_\alpha) \to 0$.

In the following theorem, we characterize functionals which are $b_{Y,X}$-continuous at 0 in the spirit of [17].

Theorem 4.1.6. *Let $f \in X^*$. The following are equivalent.*

(*a*) *f is $b_{Y,X}$-continuous at* 0.

(*b*) *For each $\varepsilon > 0$, there are w^*-slices $S_1, S_2, \cdots, S_n$ of X_1^* determined by elements of Y and a function $F : \prod S_i \to X^*$ such that $F(f_1, f_2, \cdots, f_n) = \sum_i^n a_i f_i$ where $a_i \in \mathbb{R}$ are dependent on $(f_1, f_2, \cdots, f_n)$ and $\|f - F(f_1, f_2, \cdots, f_n)\| \leq \varepsilon$ for all $(f_1, f_2, \cdots, f_n) \in \prod S_i$.*

Proof. $(a) \Rightarrow (b)$. We assume $\|f\| = 1$. Let $\varepsilon > 0$ and $A = \{x \in X_1 : f(x) \geq \varepsilon\}$. Then $0 \notin A$. Thus by definition of $b_{Y,X}$-continuity of f at 0, we have closed balls $B_X[y_i, r_i]$ with $y_i \in Y$, $i = 1, 2, \ldots, n$, such that $A \subseteq \cup_{i=1}^n B_X[y_i, r_i]$ and $0 \notin \cup_{i=1}^n B_X[y_i, r_i]$.

Let $S_i = \{g \in X_1^* : g(y_i) > r_i\}$.

Fix $(f_1, f_2, \cdots, f_n) \in \prod S_i$. Since $\inf f_i(B_X[y_i, r_i]) = f_i(y_i) - r_i\|f_i\| \geq f_i(y_i) - r_i > 0$, it follows that $A \subseteq \cup_{i=1}^n \{x \in A : f_i(x) > 0\}$. Without loss of generality we may assume that $\{1, 2, \ldots, n\}$ is the smallest set such that

$$A \subseteq \cup_{i=1}^n \{x \in A : f_i(x) > 0\}.$$

Take $H = \cap_i^n \ker f_i$. Then $A \cap H = \emptyset$. Hence $\|f|_H\| \leq \varepsilon$. Let $h \in X^*$ be a Hahn-Banach extension of $f|_H$. Then $f - h = \sum_i^n a_i f_i$ for some $a_i \in \mathbb{R}$ and

$$\|f - \sum_i^n a_i f_i\| = \|h\| = \|f|_H\| \leq \varepsilon.$$

Define
$$F(f_1, f_2, \cdots, f_n) = \sum_i^n a_i f_i.$$

$(b) \Rightarrow (a)$. It suffices to show that if $A \subseteq X$ is bounded and $\inf f(A) > \varepsilon$ for some $\varepsilon > 0$ then there exist finitely many closed balls $B_X[y_i, r_i]$ with $y_i \in Y$, $i = 1, \ldots n$, such that $A \subseteq \cup_{i=1}^n B_X[y_i, r_i]$ and $0 \notin \cup_{i=1}^n B_X[y_i, r_i]$.

Let $M = \sup_{x \in A} \|x\|$. There exists w*-slices $S_1, S_2, \cdots, S_n$ of X_1^* determined by elements of Y and $F : \prod S_i \to X^*$ such that $F(f_1, f_2, \cdots, f_n) = \sum_i^n a_i f_i$ where $a_i \in \mathbb{R}$ are dependent on $(f_1, f_2, \cdots, f_n)$ and $\|f - F(f_1, f_2, \cdots, f_n)\| \leq \varepsilon/2M$ for all $(f_1, f_2, \cdots, f_n) \in \prod S_i$.

For any $x \in A$ and $g \in F(\prod S_i)$, we have
$$g(x) = f(x) - (f - g)(x) \geq \varepsilon - \|f - g\|\|x\| \geq \varepsilon - \frac{\varepsilon}{2M} M = \frac{\varepsilon}{2}.$$

Hence
$$\inf g(A) \geq \frac{\varepsilon}{2} \text{ for all } g \in F(\prod S_i). \tag{4.1}$$

Let $A_i = \{x \in A : g(x) \neq 0 \text{ for all } g \in S_i\}$. If there is an $x \in A \setminus \cup_i^n A_i$, then there exists $f_i \in S_i$ such that $f_i(x) = 0$ and hence $F(f_1, f_2, \cdots f_n)(x) = 0$ which contradicts (4.1). Thus $A \subseteq \cup_i^n A_i$.

For each $i = 1, \ldots n$, write
$$\begin{aligned} A_i^+ &= \{x \in A_i : g(x) > 0 \text{ for all } g \in S_i\} \quad \text{and} \\ A_i^- &= \{x \in A_i : g(x) < 0 \text{ for all } g \in S_i\}. \end{aligned}$$

Clearly, $A_i = A_i^+ \cup A_i^-$. Now observe that
$$S_i \subseteq \{g \in X_1^* : g(x) > 0 \text{ for all } x \in A_i^+\}.$$

Thus by Theorem 4.1.4, there exists a closed ball B_i^+ with centre in Y such that $A_i^+ \subseteq B_i^+$ and $0 \notin B_i^+$. Similarly, there is a closed ball B_i^- with centre in Y such that $-A_i^- \subseteq B_i^-$ and $0 \notin B_i^+$. Thus we have,
$$A = \cup_i^n A_i = \cup_i^n (A_i^+ \cup A_i^-) \subseteq \cup_i^n (B_i^+ \cup -B_i^-)$$

and $0 \notin \cup_i^n (B_i^+ \cup -B_i^-)$. This completes the proof. $\square$

In the following lemma, we characterize $b_{Y,X}$ convergence at 0. The proof is similar to [36].

Lemma 4.1.7. *Let Y be a subspace of a Banach space X. Let $\{x_\alpha\}$ be a net in X_1. Then $x_\alpha \to 0$ in $b_{Y,X}$ topology if and only if $\lim d(y, \mathbb{R}x_\alpha) = 1$ for every unit vector $y \in Y$.*

Proof. Let $\lim d(y, \mathbb{R}x_\alpha) = 1$ for every unit vector $y \in Y$.

Let $B_X[y, r] \subseteq X \setminus \{0\}$ be a closed ball with centre $y \in Y$. Now by the given condition, $\lim d(y/\|y\|, \mathbb{R}x_\alpha) = 1$, that is, $\lim d(y, \mathbb{R}x_\alpha) = \|y\|$. Since $r < \|y\|$, there exists α_0 such that for every $\alpha \geq \alpha_0$, $x_\alpha \notin B_X[y, r]$. Hence $x_\alpha \to 0$ in $b_{Y,X}$ topology.

Conversely, let $x_\alpha \to 0$ in $b_{Y,X}$ topology. We show that $\lim d(y, \mathbb{R}x_\alpha) = 1$ for every unit vector $y \in Y$.

If not, there exists a unit vector $y \in Y$, such that $\lim d(y, \mathbb{R}x_\alpha) \neq 1$. Without loss of generality, we assume that $d(y, \mathbb{R}x_\alpha) \leq r < 1$ for all α. Hence there exists $\lambda_\alpha \in [1 - r, 2]$ such that either $\lambda_\alpha x_\alpha \in B_X[y, r]$ or $-\lambda_\alpha x_\alpha \in B_X[y, r]$. Passing to subnet if necessary, we assume $\lambda_\alpha x_\alpha \in B_X[y, r]$ and $\lim \lambda_\alpha = \lambda_0 \in [1 - r, 2]$. Thus if $0 < \varepsilon < 1 - r$, we can choose α_0 such that $\lambda_0 x_\alpha \in B_X[y, r]$ for all $\alpha > \alpha_0$. Hence $x_\alpha \in B_X[y/\lambda_0, (r + \varepsilon)/\lambda_0] \subseteq X \setminus \{0\}$ for $\alpha > \alpha_0$. This contradicts $x_\alpha \to 0$ in $b_{Y,X}$ topology. $\square$

For the analogue of BGP in the subspace situation, we define,

Definition 4.1.8. Let Y be a subspace of a Banach space X. We say X has the property $B_{Y,X}$ if every $f \in X^*$ is $b_{Y,X}$ continuous at 0.

In the following proposition we characterize the property $B_{Y,X}$.

Theorem 4.1.9. *Let Y be a subspace of a Banach space X. Then the following are equivalent :*

(*a*) *X has $B_{Y,X}$.*

(*b*) *Given any closed bounded convex set $A \subseteq X_1$ and $y \in Y$ such that $y \notin A$, we have finitely many closed balls $B_1, B_2, \cdots, B_n$ in X with centres in Y, such that $A \subseteq \cup_i^n B_i$ and $y \notin \cup_i^n B_i$.*

(*c*) *If a net* $\{x_\alpha\} \subseteq X_1$ *is such that* $\lim d(y, \mathbb{R}x_\alpha) = 1$ *for every unit vector* $y \in Y$, *then* $\{x_\alpha\}$ *is weakly null.*

Proof. The equivalence of (a) and (b) follows from the definition of $b_{Y,X}$-continuity and the fact that this topology is invariant under translation by elements of Y.

By Lemma 4.1.7, (c) is equivalent to : If $\{x_\alpha\} \subseteq X_1$ and $x_\alpha \to 0$ in $b_{Y,X}$-topology, then $\{x_\alpha\}$ is weakly null. Thus, every $x^* \in X^*$ is $b_{Y,X}$-continuous at 0 and vice versa. □

It was proved in [32] that BGP implies nicely smooth. Here we show that if X has $B_{Y,X}$ then Y is a VN-subspace of X. Recall from Chapter 1, $\mathcal{N} = \{F : F$ is a w*-closed subspace of X^* and $F|_Y$ is a norming subspace for $Y\}$ and $N = \cap\mathcal{N}$.

Proposition 4.1.10. *Let* $f \in X^*$ *is* $b_{Y,X}$*-continuous at* 0. *Then* $f \in N$.

Proof. We show that $f \in F$ for every $F \in \mathcal{N}$.

If not, choose $x \in F_\perp$ such that $f(x) > \varepsilon > 0$ for some ε. Since f is $b_{Y,X}$-continuous at 0, we have a closed ball $B_X[y, r]$ in X with centre in Y such that $x \in B_X[y, r]$ and $0 \notin B_X[y, r]$. Thus $\|x - y\| \leq r \leq \|y\|$.

But by Lemma 2.4.1, $F_\perp \subseteq O(Y, X)$ and hence $\|x - y\| \geq \|y\|$ for all $y \in Y$. A contradiction. □

Corollary 4.1.11. *If* X *has* $B_{Y,X}$ *then* Y *is a* VN*-subspace of* X.

Chapter 5

Weighted Chebyshev centres and intersection properties of balls in Banach spaces

In this chapter, we study weighted Chebyshev centres and their relationship with intersection properties of balls. We extend some earlier results in a more general set-up and relate them with notion of minimal points. Our results lead to a characterization of L^1-predual spaces.

5.1 General results

We start by extending Theorem 1.1.5. We need the following notions.

Definition 5.1.1. Let X be a Banach space and $A \subseteq X$.

(a) We define a partial ordering on X as follows : for $x_1, x_2 \in X$, we say that $x_1 \leq_A x_2$ if $\|x_1 - a\| \leq \|x_2 - a\|$ for all $a \in A$.

Note that $\leq_A$ defines a partial order on any Banach space containing A and we will use the same notation in all such cases.

(b) Let Y be a subspace of X and $A \subseteq Y$. Following [35], we say $x \in X$ is a minimal point of A with respect to Y if for any $y \in Y$, $y \leq_A x$

implies $y = x$. We denote the set of all minimal points of A with respect to Y in X by $A_{Y,X}$. Note $A_{Y,X} \supseteq A$.

For $A \subseteq X$, the set $A_{X,X}$ is simply the set of points of X that are minimal with respect to the ordering $\leq_A$. We will denote it by $\min A$ and call its elements minimal points of A.

(c) A function $f : X \to \mathbb{R}^+$ is said to be A-monotone if $f(x_1) \leq f(x_2)$ whenever $x_1 \leq_A x_2$.

Definition 5.1.2. Let Y be a subspace of a Banach space X. Let $\mathcal{A}$ be a family of subsets of Y.

(a) We say that Y is an almost $\mathcal{A}$-C-subspace of X if for every $A \in \mathcal{A}$, $x \in X$ and $\varepsilon > 0$, there exists $y \in Y$ such that

$$\|y - a\| \leq \|x - a\| + \varepsilon \text{ for all } a \in A. \tag{5.1}$$

(b) We say that Y is an $\mathcal{A}$-C-subspace of X if we can take $\varepsilon = 0$ in (a).

(c) If $\mathcal{A}$ is a family of subsets of X, we say that X has the (almost) $\mathcal{A}$-IP if X is an (almost) $\mathcal{A}$-C-subspace of X^{**}.

We would like to give names to the following special families.

(i) $\mathcal{F}$ = the family of all finite sets,

(ii) $\mathcal{K}$ = the family of all compact sets,

(iii) $\mathcal{B}$ = the family of all bounded sets,

(iv) $\mathcal{P}$ = the power set.

Since these families depend on the space in which they are considered, we will use the notation $\mathcal{F}(X)$ etc. whenever there is scope for confusion.

Remark 5.1.3. (a) Notice that a $\mathcal{P}$-C-subspace is just an AC-subspace, discussed in Chapter 2, $\mathcal{P}$-IP is just $IP_{f,\infty}$ and we will stick to the earlier terminology. Spaces with $\mathcal{F}$-IP were said to belong to the class GC in [63]. $\mathcal{F}$-C subspaces were called central (C-) subspaces in [10].

(b) The definition of an almost $\mathcal{A}$-C-subspace is adapted from that of almost central subspace defined in [58]. The exact analogue of the definition in [58] would have, in place of condition (5.1),

$$\sup_{a\in A}\|y-a\| \leq \sup_{a\in A}\|x-a\| + \varepsilon. \tag{5.2}$$

Clearly, our condition is stronger. We observe below (see Proposition 5.1.7) that this definition is more natural in our context.

(c) By Lemma 1.2.7, if Y is an ideal in X then Y is an almost $\mathcal{F}$-C-subspace of X. In particular, any Banach space has the almost $\mathcal{F}$-IP.

The following theorem extends Theorem 1.1.5 and is the main result of this section.

Theorem 5.1.4. *For a Banach space X and a family $\mathcal{A}$ of bounded subsets of X, the following are equivalent :*

(a) *X has the $\mathcal{A}$-IP.*

(b) *For every $A \in \mathcal{A}$ and every $f : X^{**} \longrightarrow \mathbb{R}_+$ that is A-monotone and w^*-lower semicontinuous (henceforth, w^*-lsc), the infimum of f over X^{**} and X are equal and is attained at a point of X.*

(c) *For every $A \in \mathcal{A}$ and every ρ, the infimum of $\phi_{A,\rho}$ over X^{**} and X are equal and is attained at a point of X.*

Moreover, the point in (b) or (c) can be chosen from $\min A$.

We need to develop some related results, some of which are also of independent interest, to prove the above theorem. The first one extends Veselý's result to bounded sets in dual spaces.

Let $A \subseteq X$ is bounded. Define,

$$\Delta(A) = \inf_{a_1\in A}\sup_{a_2\in A}\|a_1 - a_2\| < \infty$$

Theorem 5.1.5. (a) *Let $A \subseteq X$ be bounded and let $x \notin \overline{A + \Delta(A)X_1}$, then there exists $y \in A$ such that $y \leq_A x$.*

(*b*) *If X is a dual space and A is bounded, then every A-monotone and w^*-lsc $f : X \to \mathbb{R}^+$ attains its minimum. In particular, for every ρ, $\phi_{A,\rho}$ attains its minimum.*

(*c*) *If X is a dual space, for every $x_0 \in X$, there is a $x_1 \in \min A$ such that $x_1 \leq_A x_0$. In particular, the minimum in* (*b*) *is attained at a point of* $\min A$.

Proof. (*a*). Let $x \notin \overline{A + \Delta(A)X_1}$. Then, there exists $\varepsilon > 0$ such that $\|x - a\| > \Delta(A) + \varepsilon$ for all $a \in A$. By definition of $\Delta(A)$, there exists $y \in A$ such that $\sup_{a \in A} \|y - a\| < \Delta(A) + \varepsilon$. Clearly, $y \leq_A x$.

(*b*). By (*a*), if $x \notin \overline{A + \Delta(A)X_1}$, there exists $y \in A$ such that $y \leq_A x$, and hence, $f(y) \leq f(x)$. Thus, the infimum of f over X equals the infimum over $\overline{A + \Delta(A)X_1}$. Moreover, since X is a dual space and f is w*-lsc, it attains its minimum over any w*-compact set. Thus f actually attains its minimum over X as well.

Since the norm on X is w*-lsc, so is $\phi_{A,\rho}$ for every ρ.

(*c*). Consider $\{x \in X : x \leq_A x_0\}$. Let $\{x_i\}$ be a totally ordered subset. Let z be a w*-limit point of x_i. Since the norm is w*-lsc, we have

$$\|z - a\| \leq \liminf \|x_i - a\| = \inf \|x_i - a\| \text{ for all } a \in A.$$

Thus the family $\{x_i\}$ is $\leq_A$-bounded below by z.

By Zorn's lemma, there is a $x_1 \in \min A$ such that $x_1 \leq_A x_0$.

Now let x_0 be a minimum for f. There is a $x_1 \in \min A$ such that $x_1 \leq_A x_0$. Clearly, f attains its minimum also at x_1. □

Remark 5.1.6. If X is a dual space, apart from $\phi_{A,\rho}$, there are many examples of A-monotone and w*-lsc $f : X \to \mathbb{R}^+$. One particular example that has been treated extensively in [12] is the function ϕ_μ defined by $\phi_\mu(x) = \int_A \|x - a\|^2 d\mu(a)$, where μ is a probability measure on a compact set $A \subseteq X$.

Proposition 5.1.7. *Let Y be a subspace of a Banach space X. Let $\mathcal{A}$ be a family of bounded subsets of Y. Then the following are equivalent :*

(*a*) *Y is an almost $\mathcal{A}$-C-subspace of X*

(*b*) *for all $A \in \mathcal{A}$ and $\rho : A \to \mathbb{R}^+$, if $\cap_{a\in A} B_X[a, \rho(a)] \neq \emptyset$, then for every $\varepsilon > 0$, $\cap_{a\in A} B_Y[a, \rho(a) + \varepsilon] \neq \emptyset$.*

(*c*) *for every bounded ρ, the infimum of $\phi_{A,\rho}$ over X and Y are equal.*

Proof. Equivalence of (a) and (b) is immediate and does not need $A \in \mathcal{A}$ to be bounded.

(a) $\Rightarrow$ (c). Let Y be an almost $\mathcal{A}$-C-subspace of X, $A \in \mathcal{A}$ and $\rho : A \to \mathbb{R}^+$ be bounded. Let $M = \sup \rho(A)$. Let $\varepsilon > 0$. By definition, for $x \in X$, there exists $y \in Y$ such that

$$\|y - a\| \leq \|x - a\| + \varepsilon \text{ for all } a \in A.$$

It follows that

$$\rho(a)\|y - a\| \leq \rho(a)\|x - a\| + \rho(a)\varepsilon \leq \rho(a)\|x - a\| + M\varepsilon \text{ for all } a \in A.$$

and hence,

$$\phi_{A,\rho}(y) \leq \phi_{A,\rho}(x) + M\varepsilon.$$

Therefore,

$$\inf \phi_{A,\rho}(Y) \leq \inf \phi_{A,\rho}(X) + M\varepsilon.$$

As ε is arbitrary, the infimum of $\phi_{A,\rho}$ over X and Y are equal.

(c) $\Rightarrow$ (a). Let $A \in \mathcal{A}$, $x \in X$ and $\varepsilon > 0$. We need to show that there exists $y \in Y$ such that

$$\|y - a\| \leq \|x - a\| + \varepsilon \text{ for all } a \in A.$$

If $x \in Y$, nothing to prove. Let $x \in X \setminus Y$. Let $N = \sup_{a\in A} \|x - a\|$. Let $\rho(a) = 1/\|x - a\|$. Since $x \notin Y$ and $A \subseteq Y$, ρ is bounded. Then $\phi_{A,\rho}(x) = 1$, and therefore, $\inf \phi_{A,\rho}(X) \leq 1$. By assumption, $\inf \phi_{A,\rho}(Y) = \inf \phi_{A,\rho}(X) \leq 1$, and so, there exists $y \in Y$, such that $\phi_{A,\rho}(y) \leq 1 + \varepsilon/N$. This implies $\|y - a\| \leq \|x - a\| + \varepsilon\|x - a\|/N \leq \|x - a\| + \varepsilon$ for all $a \in A$. $\square$

Lemma 5.1.8. *Let Y be a subspace of a Banach space X. For $A \subseteq Y$, the following are equivalent :*

(a) *For every A-monotone $f : A \longrightarrow \mathbb{R}_+$ and $x \in X$, there exists $y \in Y$ such that $f(y) \leq f(x)$.*

(b) *For every $\rho : A \longrightarrow \mathbb{R}_+$ and $x \in X$, there exists $y \in Y$ such that $\phi_{A,\rho}(y) \leq \phi_{A,\rho}(x)$.*

(c) *For every continuous $\rho : A \longrightarrow \mathbb{R}_+$ and $x \in X$, there exists $y \in Y$ such that $\phi_{A,\rho}(y) \leq \phi_{A,\rho}(x)$.*

(d) *For every bounded $\rho : A \longrightarrow \mathbb{R}_+$ and $x \in X$, there exists $y \in Y$ such that $\phi_{A,\rho}(y) \leq \phi_{A,\rho}(x)$.*

(e) *Any family of closed balls centred at points of A that intersects in X also intersects in Y.*

(f) *for any $x \in X$, there exists $y \in Y$ such that $y \leq_A x$.*

(g) *$A_{Y,X} \subseteq Y$.*

It follows that whenever any of the above conditions is satisfied, for every A-monotone $f : A \longrightarrow \mathbb{R}_+$, the infimum of f over X and Y are equal and if A has a weighted Chebyshev centre in X, it has one in Y too.

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c), (b) $\Rightarrow$ (d), (f) $\Rightarrow$ (g) and (e) $\Leftrightarrow$ (f) $\Rightarrow$ (a) are obvious.

(c) or (d) $\Rightarrow$ (f). As in the proof of Proposition 5.1.7, assuming $x \notin Y$, let $\rho(a) = 1/\|x - a\|$. Then ρ is continuous and bounded and $\phi_{A,\rho}(x) = 1$. Thus, there exists $y \in Y$ such that $\phi_{A,\rho}(y) \leq 1$. This implies $\|y - a\| \leq \|x - a\|$ for all $a \in A$.

(g) $\Rightarrow$ (f) Suppose $A \in \mathcal{A}$ and $x \in X \setminus Y$. If there is no $y \in Y$ such that $\|y - a\| \leq \|x - a\|$ for all $a \in A$, then $x \in A_{Y,X}$. But this contradicts $A_{Y,X} \subseteq Y$. □

Remark 5.1.9. Let $\mathcal{A}$ be a family of bounded subsets of Y. Then Y is an $\mathcal{A}$-C-subspace of X if and only if any, and hence all, of the conditions of Theorem 5.1.8 holds for all $A \in \mathcal{A}$.

Proof of Theorem 5.1.4. (a) $\Rightarrow$ (b). By Theorem 5.1.5(b), it follows that f attains its minimum over X^{**}. Since X has $\mathcal{A}$-IP, we have the conclusion from Lemma 5.1.8.

$(b) \Rightarrow (c)$ is trivial.

$(c) \Rightarrow (a)$. Follows from equivalence of (b) and (e) in Lemma 5.1.8.

That the points in (b) and (c) can be chosen from $\min A$ follows from Theorem 5.1.5(b). $\square$

5.2 Some more results on $\mathcal{A}$-C-subspaces

In this section we compile several interesting facts about $\mathcal{A}$-C-subspaces and the $\mathcal{A}$-IP, no longer restricting ourselves to bounded sets.

Example 5.2.1. Since any Banach space X has the almost $\mathcal{F}$-IP, it follows from Proposition 5.1.7 that for any finite set $A \subseteq X$, the infimum of $\phi_{A,\rho}$ over X and X^{**} are the same [63, Theorem 2.6]. However, if A is infinite, the infimum of $\phi_{A,\rho}$ over X and X^{**} may not be the same. This was also shown by Veselý [63]. His example is $X = c_0$, $A = \{e_n : n \geq 1\}$ is the canonical unit vector basis of c_0 and $\rho \equiv 1$. Then $\inf \phi_{A,\rho}(X) = 1$ and $\inf \phi_{A,\rho}(X^{**}) = 1/2$. The example clearly also excludes countable, bounded, or, taking $A \cup \{0\}$, even weakly compact sets. Thus by Proposition 5.1.7, c_0 fails the almost $\mathcal{B}$-IP, almost $\mathcal{P}$-IP and if $\mathcal{A}$ is the family of countable or weakly compact sets, then c_0 fails the almost $\mathcal{A}$-IP too.

However, the equality holds also for compact sets.

Lemma 5.2.2. *Let $\mathcal{A}$ and $\mathcal{A}_1$ be two families of subsets of Y such that for every $A \in \mathcal{A}$ and $\varepsilon > 0$, there exists $A_1 \in \mathcal{A}_1$ such that $A \subseteq A_1 + \varepsilon Y_1$. If Y is an almost $\mathcal{A}_1$-C-subspace of X, then Y is an almost $\mathcal{A}$-C-subspace of X as well.*

Proof. Let $A \in \mathcal{A}$ and $\varepsilon > 0$. By hypothesis, there exists $A_1 \in \mathcal{A}_1$ such that $A \subseteq A_1 + \varepsilon/3Y_1$. Let $x \in X$. Since Y is an almost $\mathcal{A}_1$-C-subspace of X, there exists $y \in Y$ such that

$$\|y - a_1\| \leq \|x - a_1\| + \varepsilon/3 \quad \text{for all } a_1 \in A_1.$$

Now fix $a \in A$. Then there exists $a_1 \in A_1$ such that $\|a - a_1\| < \varepsilon/3$. Then

$$\begin{aligned} \|y-a\| &\leq \|y-a_1\| + \|a-a_1\| \leq \|x-a_1\| + 2\varepsilon/3 \\ &\leq \|x-a\| + \|a-a_1\| + 2\varepsilon/3 \leq \|x-a\| + \varepsilon. \end{aligned}$$

Therefore, Y is an almost $\mathcal{A}$-C-subspace of X as well. □

Proposition 5.2.3. (a) *Any ideal is an almost $\mathcal{K}$-C-subspace.*

(b) *Any Banach space has the almost $\mathcal{K}$-IP.*

(c) *If A is a compact subset of X and $\rho : A \longrightarrow \mathbb{R}_+$ is bounded, then the infimum of $\phi_{A,\rho}$ over X and X^{**} are the same.*

Proof. Since any ideal is an almost $\mathcal{F}$-C-subspace, and any Banach space has the almost $\mathcal{F}$-IP, (a) and (b) follows from the above lemma. Now (c) follows from Proposition 5.1.7. □

Proposition 5.2.4. (a) *Let Y be a subspace of a Banach space X. Let $\mathcal{A}$ be a family of subsets of Y and let $\mathcal{A}_1$ be a subfamily of $\mathcal{A}$. If Y is a $\mathcal{A}$-C-subspace of X, then Y is a $\mathcal{A}_1$-C-subspace of X as well. In particular, $IP_{f,\infty}$ implies $\mathcal{B}$-IP implies $\mathcal{K}$-IP implies $\mathcal{F}$-IP.*

(b) *Constrained subspaces are $\mathcal{A}$-C-subspaces for any $\mathcal{A}$.*

(c) *Let $Z \subseteq Y \subseteq X$ and let $\mathcal{A}$ be a family of subsets of Z. If Z is an $\mathcal{A}$-C-subspace of X, then Z is an $\mathcal{A}$-C-subspace of Y. And, if Y is an $\mathcal{A}$-C-subspace of X, then the converse also holds.*

Proof. Follows from definition. □

Proposition 5.2.5. *For a family $\mathcal{A}$ of subsets of a Banach space X, the following are equivalent :*

(a) *X has the $\mathcal{A}$-IP*

(b) *X is a $\mathcal{A}$-C-subspace of some dual space.*

(c) *for all $A \in \mathcal{A}$ and $\rho : A \to \mathbb{R}^+$, $\cap_{i=1}^n B_X[a_i, \rho(a_i) + \varepsilon] \neq \emptyset$ for all finite subset $\{a_1, a_2, \ldots, a_n\} \subseteq A$ and for all $\varepsilon > 0$ implies $\cap_{a \in A} B_X[a, \rho(a)] \neq \emptyset$.*

In particular, any dual space has the $\mathcal{A}$-IP for any $\mathcal{A}$. Let $\mathcal{S}$ be any of the families $\mathcal{F}$, $\mathcal{K}$, $\mathcal{B}$ or $\mathcal{P}$. The $\mathcal{S}$-IP is inherited by $\mathcal{S}$-C-subspaces, in particular, by constrained subspaces.

Proof. Clearly, $(a) \Rightarrow (b)$, while $(c) \Rightarrow (a)$ follows from the Lemma 1.2.7.

$(b) \Rightarrow (c)$. Let X be an $\mathcal{A}$-C-subspace of Z^*. Consider the family $\{B_{Z^*}[a, \rho(a) + \varepsilon] : a \in A, \varepsilon > 0\}$ in Z^*. Then, by the hypothesis, any finite subfamily intersects. Hence, by w*-compactness, $\cap_{a\in A} B_{Z^*}[a, \rho(a)] \neq \emptyset$. Since X is an $\mathcal{A}$-C-subspace of Z^*, we have $\cap_{a\in A} B_X[a, \rho(a)] \neq \emptyset$. □

The following result significantly improves [10, Proposition 2.8] and provides yet another characterization of the $\mathcal{A}$-IP.

Proposition 5.2.6. *Let Y be an almost $\mathcal{F}$-C subspace of (in particular, an ideal in) a Banach space X. Let $\mathcal{A}$ be a family of subsets of Y. If Y has the $\mathcal{A}$-IP, then Y is an $\mathcal{A}$-C-subspace of X.*

Proof. Let $x \in X$, $A \in \mathcal{A}$. Since Y be an almost $\mathcal{F}$-C subspace of X, for all finite subset $\{a_1, a_2, \ldots, a_n\} \subseteq A$ and for all $\varepsilon > 0$, $\cap_{i=1}^n B_Y[a_i, \|x - a_i\| + \varepsilon] \neq \emptyset$. Since Y has the $\mathcal{A}$-IP, by Proposition 5.2.5(c), $\cap_{a\in A} B_Y[a, \|x - a\|] \neq \emptyset$. □

Since X is always an ideal in X^{**}, the following corollary is immediate.

Corollary 5.2.7. *For a Banach space X and a family $\mathcal{A}$ of subsets of X, the following are equivalent :*

(a) *X has $\mathcal{A}$-IP.*

(b) *X is an $\mathcal{A}$-C-subspace of every superspace Z in which X embeds as an almost $\mathcal{F}$-C subspace.*

(c) *X is an $\mathcal{A}$-C-subspace of every superspace Z in which X embeds as an ideal.*

5.3 Strict convexity and minimal points

Our first result extends [63, Theorem 32] and addresses the uniqueness question for Chebyshev centres of compact sets.

Proposition 5.3.1. *Let A be a compact subset of a strictly convex Banach space X. Then for each continuous ρ, A admits at most one weighted Chebyshev centre.*

Proof. Suppose A admits two distinct weighted Chebyshev centres $x_0, x_1 \in X$. Then $\phi_{A,\rho}(x_0) = \phi_{A,\rho}(x_1) = r$ (say). Then for all $a \in A$, we have x_1, $x_0 \in B_X[a, r/\rho(a)]$. By strict convexity, $z = (x_1 + x_0)/2$ is in the interior of $B_X[a, r/\rho(a)]$ for all $a \in A$. Thus, $\rho(a)\|z - a\| < r$, for all $a \in A$. Since ρ is continuous, $\phi_{A,\rho}(z) < r$, which contradicts that minimum value is r. ☐

Our main result in this section is the following :

Theorem 5.3.2. *Let X be a Banach space such that*

(*i*) *X has the $\mathcal{F}$-IP; and*

(*ii*) *for every compact set $A \subseteq X$,* $\min A$ *is weakly compact.*

Then X has the $\mathcal{K}$-IP. Moreover, if X^{**} is strictly convex, then the converse also holds.

Proof. Let X have the $\mathcal{F}$-IP and for every compact set $A \subseteq X$, let $\min A$ be weakly compact. Observe that for any $B \subseteq A$, we have $\min B \subseteq \min A$.

Let $A \subseteq X$ be compact and let $x^{**} \in X^{**}$. By Lemma 5.1.8, it suffices to show that there is a $z_0 \in X$ such that $\|z_0 - a\| \leq \|x^{**} - a\|$ for all $a \in A$.

Let $\{a_n\}$ be norm dense in A. Let $\varepsilon_k > 0$, $\varepsilon_k \to 0$. By compactness of A, for each k, there is a n_k such that $A \subseteq \bigcup_1^{n_k} B_X[a_n, \varepsilon_k]$. Since X has the $\mathcal{F}$-IP, there exists $z_k \in \cap_1^{n_k} B_X[a_n, \|x^{**} - a_n\|]$ and $z_k \in \min\{a_1, a_2 \cdots a_{n_k}\} \subseteq \min A$. Then $\|z_k - a\| \leq \|x^{**} - a\| + 2\varepsilon_k$ for all $a \in A$. Now, by weak compactness of $\min A$, we have, by passing to a subsequence if necessary, $z_k \to z_0$ weakly for some $z_0 \in X$. Since the norm is weakly lsc, we have $\|z_0 - a\| \leq \liminf \|z_k - a\| \leq \|x^{**} - a\|$ for all $a \in A$.

Conversely, let X have the $\mathcal{K}$-IP and X^{**} be strictly convex. Let $A \subseteq X$ be compact. It is enough to show that any sequence $\{x_n\} \subseteq \min A$ has a weakly convergent subsequence. Without loss of generality, we may assume that $\{x_n\}$ are all distinct. By Theorem 5.1.5 (a), $\min A \subseteq \overline{A + \Delta(A)X_1}$ is bounded. Let x^{**} be a w*-cluster point of $\{x_n\}$ in X^{**}. It suffices to show that $x^{**} \in X$.

Suppose $x^{**} \in X^{**} \setminus X$. Since X has the $\mathcal{K}$-IP, there exists $x_0 \in \min A$ such that $\|x_0 - a\| \leq \|x^{**} - a\|$ for all $a \in A$. Since X^{**} is strictly convex, $\|(x^{**}+x_0)/2-a\| < \|x^{**}-a\|$ for all $a \in A$. Since $(x^{**}+x_0)/2 \in X^{**}\setminus X$, by $\mathcal{K}$-IP again, there exists $z_0 \in \min A$ such that $\|z_0-a\| \leq \|(x^{**}+x_0)/2-a\| < \|x^{**} - a\|$ for all $a \in A$.

Since A is compact, there exists $\varepsilon > 0$ such that $\|z_0 - a\| < \|x^{**} - a\| - \varepsilon$ for all $a \in A$. Observe that

$$\|z_0 - a\| < \|x^{**} - a\| - \varepsilon \leq \liminf_n \|x_n - a\| - \varepsilon \text{ for all } a \in A.$$

Therefore, for every $a \in A$, there exists $N(a) \in \mathbb{N}$ such that for all $n \geq N(a)$, $\|z_0 - a\| < \|x_n - a\| - \varepsilon$. By compactness, there exists $N \in \mathbb{N}$ such that $\|z_0-a\| < \|x_n-a\|-\varepsilon/4$ for all $n \geq N$ and $a \in A$. Thus, $z_0 \leq_A x_n$ for all $n \geq N$. Since $x_n \in \min A$, $z_0 = x_n$ for all $n \geq N$. This contradiction completes the proof. □

Remark 5.3.3. In proving sufficiency, one only needs that $\{z_k\}$ has a subsequence convergent in a topology in which the norm is lsc. The weakest such topology is the ball topology, b_X. So it follows that if X has the $\mathcal{F}$-IP and for every compact set $A \subseteq X$, $\min A$ is b_X-compact, then X has the $\mathcal{K}$-IP. Is the converse true?

Corollary 5.3.4. *[12, Corollary 1] Let X be a reflexive and strictly convex Banach space. Let $A \subseteq X$ be a compact set. Then* $\min A$ *is weakly compact.*

Remark 5.3.5. Clearly, our proof is simpler than the original proof of [12].

If Z is a non-reflexive Banach space with Z^{***} strictly convex, then $X = Z^*$ is a non-reflexive Banach space with $\mathcal{K}$-IP such that X^{**} is strictly convex. Thus, our result is also stronger than [12, Corollary 1].

5.4 L_1-preduals and $\mathcal{P}_1$-spaces

Our next theorem extends [10, Theorem 7], exhibits a large class of Banach spaces with the $\mathcal{K}$-IP and produces a family of examples where the notions of $\mathcal{F}$-C-subspaces and $\mathcal{K}$-C-subspaces are equivalent.

Theorem 5.4.1. *For a Banach space X, the following are equivalent :*

(a) *X is a $\mathcal{K}$-C-subspace of every superspace*

(b) *X is a $\mathcal{K}$-C-subspace of every dual superspace*

(c) *X is a $\mathcal{F}$-C-subspace of every superspace*

(d) *X is a $\mathcal{F}$-C-subspace of every dual superspace*

(e) *X is an almost $\mathcal{F}$-C-subspace of every superspace*

(f) *X is an almost $\mathcal{F}$-C-subspace of every dual superspace*

(g) *X is an L^1-predual.*

Proof. Observe that if $X \subseteq Y \subseteq Y^{**}$ and X is a $\mathcal{A}$-C-subspace of Y^{**}, then X is a $\mathcal{A}$-C-subspace of Y. Thus $(a) \Leftrightarrow (b)$ and $(c) \Leftrightarrow (d)$. And clearly, $(a) \Rightarrow (c) \Rightarrow (e) \Rightarrow (f)$.

$(f) \Rightarrow (g)$. Consider X embedded isometrically in $\ell_\infty(X_1^*)$. Suppose $\{B_X[x_i, r]\}$ is a finite family of pairwise intersecting balls in X and $\varepsilon > 0$ be given. Since $\ell_\infty(X_1^*)$ is a dual L^1-predual space, the balls intersects in $\ell_\infty(X_1^*)$. By (f), X is an almost $\mathcal{F}$-C-subspace of $\ell_\infty(X_1^*)$, thus $\cap B_X[x_i, r + \varepsilon] \neq \emptyset$. The result follows from Proposition 1.2.10(e).

$(g) \Rightarrow (a)$. Suppose X is an L^1-predual, and let $X \subseteq Z$. Let $A \subseteq X$ be compact with at least three points. Let $z_0 \in Z$. Then the family of balls $\{B_X[a, \|z_0 - a\|] : a \in A\}$ have the weak intersection property. Since X is an L^1-predual and since the centres of the balls are in a compact set, by Proposition 1.2.10(d), $\cap_{a\in A} B_X[a, \|z_0 - a\|] \neq \emptyset$.

If A has two points, observe that two balls intersect if and only if the distance between the centres is less than or equal to the sum of the radii, it is independent of the ambient space. $\square$

Corollary 5.4.2. *Every L^1-predual has the $\mathcal{K}$-IP and the $\mathcal{F}$-IP.*

Proposition 5.4.3. *Suppose X is an L_1-predual space. Then for a subspace $Y \subseteq X$, the following are equivalent :*

(*a*) *Y is an ideal in X*

(*b*) *Y is a $\mathcal{K}$-C-subspace of X*

(*c*) *Y is a $\mathcal{F}$-C-subspace of X*

(*d*) *Y is an almost $\mathcal{F}$-C-subspace of X*

(*e*) *Y itself is an L_1-predual*

Proof. $(e) \Rightarrow (b)$ follows from Theorem 5.4.1 and $(e) \Rightarrow (a)$ follows from Proposition 1.2.10(e). And clearly, $(b) \Rightarrow (c) \Rightarrow (d)$ and $(a) \Rightarrow (d)$.

$(d) \Rightarrow (e)$. By Proposition 1.2.10(c), it follows that Y is an L_1-predual space if and only if for every $r > 0$ and for any finite collection of pairwise intersecting balls $\{B_Y[y_i, r]\}$ and $\varepsilon > 0$, we have $\cap B_Y[y_i, r + \varepsilon] \neq \emptyset$. Now given such a family, consider the balls $\{B_X[y_i, r]\}$ in X. X being an L_1-predual, we have, given any $\varepsilon > 0, \cap B_X[y_i, r + \varepsilon] \neq \emptyset$. Since Y is an almost $\mathcal{F}$-C subspace of X, the proof follows. □

The analog of Theorem 5.4.1 for AC-subspaces involves $\mathcal{P}_1$-spaces.

Theorem 5.4.4. *For a Banach space X, the following are equivalent :*

(*a*) *X is a $\mathcal{P}_1$-space*

(*b*) *X is constrained in every dual superspace*

(*c*) *X is a AC-subspace of every superspace*

(*d*) *X is a AC-subspace of every dual superspace*

(*e*) *X is an L^1-predual with $IP_{f,\infty}$.*

Proof. $(a) \Leftrightarrow (b)$ and $(c) \Leftrightarrow (d)$ follow as in the first paragraph of Theorem 5.4.1. And clearly, $(a) \Rightarrow (c)$.

$(d) \Rightarrow (e)$. Follows from Proposition 5.2.5 and Theorem 5.4.1.

$(e) \Rightarrow (a)$. Recall that [42, Theorem 3.8] a Banach space X is a $\mathcal{P}_1$-space if and only if every pairwise intersecting family of closed balls in X

intersects. And that X is a L^1-predual if and only if X^{**} is a $\mathcal{P}_1$-space (Proposition 1.2.10(b).

Now given a pairwise intersecting family of closed balls in X, since X^{**} is a $\mathcal{P}_1$-space, they intersect in X^{**}. And since X has $IP_{f,\infty}$, they intersect in X too. $\square$

More generally, we have

Proposition 5.4.5. *Let $\mathcal{A}$ be a family of subsets of X such that $\mathcal{F} \subseteq \mathcal{A}$. Then, the following are equivalent :*

(a) *X is an L_1-predual with $\mathcal{A}$-IP*

(b) *X is an $\mathcal{A}$-C-subspace of every superspace*

(c) *for every $A \in \mathcal{A}$, every pairwise intersecting family of closed balls in X with centres in A intersects.*

Proof. (a) $\Rightarrow$ (b). Since X has the $\mathcal{A}$-IP, it is an $\mathcal{A}$-C-subspace of every superspace in which it is an ideal (Proposition 5.2.7) and since X is an L_1-predual, by Proposition 1.2.10(e), it is an ideal in every superspace. Thus (b) follows.

(b) $\Rightarrow$ (a). Since $\mathcal{F} \subseteq \mathcal{A}$, this is immediate.

(a) $\Rightarrow$ (c). This is similar to the proof of Theorem 5.4.4 (e) $\Rightarrow$ (a).

(c) $\Rightarrow$ (a). If every finite family of pairwise intersecting closed balls in X intersects, then X is an L_1-predual (Proposition 1.2.10(e)). And that X has the $\mathcal{A}$-IP follows from Proposition 5.2.5 (c). $\square$

We now characterize when $C(T, X)$ is a real L_1-predual. First we need the following lemma.

Lemma 5.4.6. *Suppose Y is a subspace of a Banach space X and Y is a real L_1-predual. Let $A \subseteq Y$ be a compact set and $r : A \to \mathbb{R}^+$ be such that $\cap_{a\in A} B_X[a, r(a)] \neq \emptyset$. Let $y \in \cap_{a\in A} B_Y[a, r(a) + \varepsilon]$ for some $\varepsilon > 0$. Then there exists $z \in \cap_{a\in A} B_Y[a, r(a)]$ such that $\|y - z\| \leq \varepsilon$.*

Proof. Since $\cap_{a\in A} B_X[a, r(a)] \neq \emptyset$, and intersection of intervals is an interval, for any $y^* \in Y_1^*$, $\cap_{a\in A} B_{\mathbb{R}}[y^*(a), r(a)] \neq \emptyset$ and is a closed interval. As $y^*(y) \in$

$\cap_{a\in A} B_{\mathbb{R}}[y^*(a), r(a)+\varepsilon]$ for any $y^* \in Y_1^*$, the family $\{B_Y[y,\varepsilon], B_Y[a,r(a)] : a \in A\}$ is a weakly intersecting family of balls in Y. Since Y is a L_1-predual, $B_Y[y,\varepsilon] \cap \cap_{a\in A} B_Y[a,r(a)] \neq \emptyset$. □

Proposition 5.4.7. *A Banach space X is a real L_1-predual if and only if for each paracompact space T, $C_b(T,X)$, the space of all bounded continuous functions on T, is a real L_1-predual.*

Proof. Since X is constrained in $C_b(T,X)$, hence a $\mathcal{K}$-C-subspace, by Proposition 5.4.3, if $C_b(T,X)$ is an L_1-predual, then so is X.

Conversely, suppose X is a real L_1-predual. Let $Z = C_b(T,X)$, $\{f_1, f_2, \ldots, f_n\} \subseteq Z$ and $r_1, r_2, \ldots, r_n > 0$ be such that the family $\{B_Z[f_i, r_i] : i = 1, \ldots, n\}$ intersects weakly. Then for each $t \in T$, the family $\{B_X[f_i(t), r_i] : i = 1, \ldots, n\}$ intersects weakly, and since X is a real L_1-predual, they intersect in X. Consider the multi-valued map $F : T \to X$ given by $F(t) = \cap_{i=1}^n B_X[f_i(t), r_i]$. Note for each t, $F(t)$ is a nonempty closed convex subset of X.

Claim : F is lower semicontinuous, that is, for each U open in X, the set $V = \{t \in T : F(t) \cap U \neq \emptyset\}$ is open in T.

Let $t_0 \in V$. Let $x_0 \in F(t_0) \cap U$. Let $\varepsilon > 0$ be such that $\|x - x_0\| < \varepsilon$ implies $x \in U$. Let W be an open subset of t_0 such that $t \in W$ implies $\|f_i(t) - f_i(t_0)\| < \varepsilon/2$ for all $i = 1, \ldots, n$. We will show that $W \subseteq V$.

Let $t \in W$. Then for any $i = 1, \ldots, n$, $\|x_0 - f_i(t)\| \leq \|x_0 - f_i(t_0)\| + \|f_i(t_0) - f_i(t)\| \leq r_i + \varepsilon/2$. Therefore, $x_0 \in \cap_{i=1}^n B_X[f_i(t), r_i + \varepsilon/2]$. By Lemma 5.4.6, there exists $z \in F(t) = \cap_{i=1}^n B_X[f_i(t), r_i]$ such that $\|x_0 - z\| \leq \varepsilon/2 < \varepsilon$. Then $z \in F(t) \cap U$, and hence, $t \in V$. This proves the claim.

Now since T is paracompact, by Michael's selection theorem (see [49, Theorem 1.1]), there exists $g \in Z$ such that $g(t) \in F(t)$ for all $t \in T$. It follows that $g \in \cap_{i=1}^n B_Z[f_i, r_i]$. □

Remark 5.4.8. For T compact Hausdorff, this result follows from [48, Corollary 2, p 43]. But our proof is simpler.

5.5 Stability results

In this section we consider some stability results. With a proof similar to [10, Proposition 14], we first observe that

Proposition 5.5.1. *$\mathcal{K}$-IP is a separably determined property, i.e., if every separable subspace of a Banach space X have $\mathcal{K}$-IP, then X also has $\mathcal{K}$-IP.*

Proof. Let every separable subspace of X has $\mathcal{K}$-IP. Let $A \subseteq X$ be a compact set and consider the function $\phi_{A,\rho}$ for any ρ. We need to show the infimum of $\phi_{A,\rho}$ is attained in X.

Let $\{x_n\}$ be a minimizing sequence for $\phi_{A,\rho}$. Consider the subspace $Y = \overline{\text{span}}(\{x_n\} \cup A)$. Y is separable and $\inf \phi_{A,\rho}(Y) = \inf \phi_{A,\rho}(X)$. But by assumption Y has $\mathcal{K}$-IP and hence $\phi_{A,\rho}$ attains its infimum over Y and so on X as well. $\square$

Definition 5.5.2. [37] A subspace Y of a Banach space X is called a semi-L-summand if there exists a (possibly nonlinear) projection $P : X \longrightarrow Y$ such that

$$\begin{aligned} P(\lambda x + Py) &= \lambda Px + Py, \text{ and} \\ \|x\| &= \|Px\| + \|x - Px\| \end{aligned}$$

for all $x, y \in X$, λ scalar.

In [10], it was shown that semi-L summands are $\mathcal{F}$-C-subspaces. Basically the same proof actually shows that

Proposition 5.5.3. *A semi-L-summand is an AC-subspace.*

Proof. Let $Y \subseteq X$ be a semi-L-summand, P be as in definition and $x \in X$. Then $\|Px - y\| \leq \|Px - y\| + \|x - Px\| = \|x - y\|$, for all y. $\square$

Our next result concerns proximinal subspaces (Definition 1.2.3).

Proposition 5.5.4. *Let $Z \subseteq Y \subseteq X$, Z proximinal in X.*

(*a*) *Let* $\mathcal{A}$ *be a family of subsets of* Y/Z. *Let* $\mathcal{A}'$ *be a family of subsets of* Y *such that for any* $x \in X$ *and* $A \in \mathcal{A}$, *there exists* $A' \in \mathcal{A}'$ *such that for any* $a + Z \in A$, $\{a + P_Z(x - a)\} \cap A' \neq \emptyset$. *Suppose* Y *is a* $\mathcal{A}'$*-C-subspace of* X. *Then* Y/Z *is a* $\mathcal{A}$*-C-subspace of* X/Z.

Let $\mathcal{S}$ *be any of the families* $\mathcal{F}$, $\mathcal{B}$ *or* $\mathcal{P}$.

(*b*) *If* Y *is a* $\mathcal{S}(Y)$*-C-subspace of* X, *then* Y/Z *is a* $\mathcal{S}(Y/Z)$*-C-subspace of* X/Z.

(*c*) *Suppose the metric projection has a continuous selection. Then, if* Y *is a* $\mathcal{K}(Y)$*-C-subspace of* X, Y/Z *is a* $\mathcal{K}(Y/Z)$*-C-subspace of* X/Z.

(*d*) *Let* $Z \subseteq Y \subseteq X^*$, Z *w*-closed in* X^*. *If* Y *is a* $\mathcal{S}(Y)$*-C-subspace of* X^*, *then* Y/Z *is a* $\mathcal{S}(Y/Z)$*-C-subspace of* X^*/Z, *and hence, has the* $\mathcal{S}(Y/Z)$*-IP.*

(*e*) *Let* X *have the* $\mathcal{S}(X)$*-IP. Let* $M \subseteq X$ *be a reflexive subspace. Then* X/M *has the* $\mathcal{S}(X/M)$*-IP.*

Proof. (*a*). Let $A \in \mathcal{A}$ and $x + Z \in X/Z$. Choose A' as above. Then, for $a + Z \in A$, there exists $z \in P_Z(x - a)$ (depending on x and a) such that $a + z \in A'$. Since Y is a $\mathcal{A}'$-C-subspace of X, there exists $y_0 \in Y$ such that $\|y_0 - a - z\| \leq \|x - a - z\|$ for all $a + Z \in A$. Clearly then $\|y_0 - a + Z\| \leq \|y_0 - a - z\| \leq \|x - a - z\| = \|x - a + Z\|$.

If $\mathcal{S}$ is the family under consideration in (*b*) and (*c*) above and $\mathcal{A} = \mathcal{S}(Y/Z)$, then for any choice of $\mathcal{A}'$ as above, $\mathcal{S}(Y) \subseteq \mathcal{A}'$. Hence, (*b*) and (*c*) follows from (*a*). For (*d*), we simply observe that any w*-closed subspace of a dual space is proximinal. And (*e*) follows from (*d*). □

As in [10, Corollary 4.6], we observe

Proposition 5.5.5. *Let* $Z \subseteq Y \subseteq X$, Z *proximinal in* Y *and* Y *is a semi-L-summand in* X. *Then* Y/Z *is a AC-subspace of* X/Z.

Let us now consider the c_0 or ℓ_p sums.

Theorem 5.5.6. *Let Γ be an index set. For all $\alpha \in \Gamma$, let Y_α be a subspace of X_α. Let X and Y denote resp. the c_0 or ℓ_p $(1 \le p \le \infty)$ sum of X_α's and Y_α's.*

(a) *For each $\alpha \in \Gamma$, let $\mathcal{A}_\alpha$ be a family of subsets of Y_α such that $\{0\} \in \mathcal{A}_\alpha$ and for any $A \in \mathcal{A}_\alpha$, there exists $B \in \mathcal{A}_\alpha$ such that $A \cup \{0\} \subseteq B$.*

Let $\mathcal{A}$ be a family of subsets of Y such that for any $\alpha \in \Gamma$, the α-section of any $A \in \mathcal{A}$ belongs to $\mathcal{A}_\alpha$.

Then Y is an $\mathcal{A}$-C-subspace of X if and only if for each $\alpha \in \Gamma$, Y_α is an $\mathcal{A}_\alpha$-C-subspace of X_α.

Let $\mathcal{S}$ be any of the families $\mathcal{F}$, $\mathcal{K}$, $\mathcal{B}$ or $\mathcal{P}$.

(b) *Y is a $\mathcal{S}(Y)$-C-subspace of X if and only if for any $\alpha \in \Gamma$, Y_α is a $\mathcal{S}(Y_\alpha)$-C-subspace of X_α.*

(c) *The $\mathcal{S}$-IP is stable under ℓ_p-sums $(1 \le p \le \infty)$.*

Proof. (a). Let Y is an $\mathcal{A}$-C-subspace of X and let $\alpha_0 \in \Gamma$. Let $x_{\alpha_0} \in X_{\alpha_0}$ and $A \in \mathcal{A}_{\alpha_0}$. Define $x \in X$ by

$$x_\alpha = \begin{cases} x_{\alpha_0} & \text{if} \quad \alpha = \alpha_0 \\ 0 & \text{otherwise} \end{cases} .$$

Similarly, for each $y_{\alpha_0} \in A$, define $y \in Y$

$$y_\alpha = \begin{cases} y_{\alpha_0} & \text{if} \quad \alpha = \alpha_0 \\ 0 & \text{otherwise} \end{cases} .$$

Call that set of all such $y \in Y$ with $y_{\alpha_0} \in A$, $\tilde{A}$. Then $\tilde{A} \in \mathcal{A}$, and there is a $y \in Y$ such that $\|y - z\| \le \|x - z\|$, for all $z \in \tilde{A}$. Easy to see that the α_0 coordinate of y works.

Conversely let for each $\alpha \in \Gamma$, Y_α be $\mathcal{A}$-C-subspace of X_α. Let $A \in \mathcal{A}$. Take $B = A \cup \{0\}$.

Let $x \in X$. For each α there is an $y_\alpha \in Y_\alpha$ such that $\|y_\alpha - z_\alpha\| \le \|x_\alpha - z_\alpha\|$, for all $z_\alpha \in B_\alpha$. Since $0 \in B$, we also have $\|y_\alpha\| \le \|x_\alpha\|$. Thus for y defined by taking α coordinate to be y_α, we have $y \in Y$ and $\|y - z\| \le \|x - z\|$, for all $z \in A$.

(c). X_α has $\mathcal{S}$-IP if and only if X_α is a $\mathcal{S}$-C-subspace of some dual space Y_α^*. Now the ℓ_p-sum $(1 \leq p \leq \infty)$ of Y_α^*'s is a dual space. □

Remark 5.5.7. The result for $\mathcal{F}$-IP was noted in [63] with a much different proof. The stability of the $IP_{f,\infty}$ under ℓ_1-sums was noted in [53] again with a different proof.

[63] also notes that $\mathcal{F}$-IP is stable under c_0-sum. And Corollary 5.4.2 shows that c_0 has the $\mathcal{K}$-IP. However, we do not know if the $\mathcal{K}$-IP is stable under c_0-sums. As for the $\mathcal{B}$-IP or $IP_{f,\infty}$, we now show that c_0-sum of any infinite family of Banach spaces lacks the $\mathcal{B}$-IP, and therefore, also the $IP_{f,\infty}$. This is quite similar to Example 5.2.1.

Proposition 5.5.8. *Let Γ be an infinite index set. For any family of Banach spaces X_α, $\alpha \in \Gamma$, $X = \oplus_{c_0} X_\alpha$ lacks the $\mathcal{B}$-IP.*

Proof. For each $\alpha \in \Gamma$, let $x_\alpha \in X_\alpha$, $\|x_\alpha\| = 1$, and define $e_\alpha \in X$ by

$$(e_\alpha)_\beta = \begin{cases} x_\alpha & \text{if} \quad \beta = \alpha \\ 0 & \text{otherwise} \end{cases}$$

Then $A = \{e_\alpha : \alpha \in \Gamma\}$ is bounded and the balls $B_{X^{**}}[e_\alpha, 1/2]$ intersect at the point $(\frac{1}{2}x_\alpha) \in X^{**}$, but the balls $B_X[e_\alpha, 1/2]$ cannot intersect in X. □

Remark 5.5.9. As before, taking $A \cup \{0\}$, it follows that X lacks the $\mathcal{A}$-IP even for $\mathcal{A} = \{$weakly compact sets$\}$.

Coming to function spaces, we note the following general result.

Proposition 5.5.10. *Let Y be a subspace of a Banach space X and $\mathcal{A}$ be a family of subsets of Y.*

(a) *For any topological space T, if $C_b(T, Y)$ is a $\mathcal{A}$-C-subspace of $C_b(T, X)$, then Y is a $\mathcal{A}$-C-subspace of X. Moreover, if $C_b(T, X)$ has $\mathcal{A}$-IP, X has $\mathcal{A}$-IP.*

(b) *Let (Ω, Σ, μ) be a probability space. If for some $1 \leq p < \infty$, $L^p(\mu, Y)$ is a $\mathcal{A}$-C-subspace of $L^p(\mu, X)$, then Y is a $\mathcal{A}$-C-subspace of X. Moreover, if $L^p(\mu, X)$, has $\mathcal{A}$-IP, then X has $\mathcal{A}$-IP.*

Proof. For both (a) and (b), let $F(X)$ denote the corresponding space of functions and identify X with the constant functions. In (a), point evaluation and in (b), integral over Ω gives a norm 1 projection from $F(X)$ onto X. Thus X inherits $\mathcal{A}$-IP from $F(X)$.

Now suppose $F(Y)$ is an $\mathcal{A}$-C-subspace of $F(X)$. Let $P : F(Y) \to Y$ be the above norm 1 projection. Let $x \in X$ and $A \in \mathcal{A}$. Then, there exists $g \in F(Y)$ such that $\|g - a\| \leq \|x - a\|$ for all $a \in A$. Let $y = Pg$. Then, $\|y - a\| \leq \|g - a\| \leq \|x - a\|$ for all $a \in A$. □

The following result was proved in [10].

Proposition 5.5.11. (a) *[10, Proposition 4.16] If X has the RNP and is constrained in Z^* for some Banach space Z, then for $1 < p < \infty$, $L^P(\mu, X)$ is constrained in $L^q(\mu, Y)^*$ $(1/p + 1/q = 1)$, and hence has the $IP_{f,\infty}$.*

(b) *[10, Proposition 4.17] Suppose X is separable and constrained in X^{**} by a projection P that is w^*-w universally measurable. Then for $1 \leq p < \infty$, $L^P(\mu, X)$ is constrained in $L^q(\mu, X^*)^*$ $(1/p + 1/q = 1)$, and hence has the $IP_{f,\infty}$.*

In [53] the author showed the following.

Proposition 5.5.12. *[53] (a) Let X be a Banach space containing c_0 and let Y be any infinite dimensional Banach space. Then c_0 is constrained in $X \otimes_\varepsilon Y$.*

(b) *Suppose X contains c_0. Then for any nonatomic measure space (Ω, Σ, μ), c_0 is constrained in $L^1(\mu, X)$.*

Proposition 5.5.13. (a) *If X is a Banach space containing c_0 and Y is an infinite dimensional Banach space, then $X \otimes_\varepsilon Y$ fails the $\mathcal{B}$-IP and $IP_{f,\infty}$.*

(b) *If $C(K, X)$ has the $\mathcal{B}$-IP, then either K is finite or X is finite dimensional.*

(c) *$C(K, X)$ has the $IP_{f,\infty}$ if and only if either (i) K is finite and X has the $IP_{f,\infty}$ or (ii) X is finite dimensional and K is extremally disconnected.*

(d) *For any nonatomic measure space (Ω, Σ, μ) and a Banach space X containing c_0, $L^1(\mu, X)$ fails the $\mathcal{B}$-IP and the $IP_{f,\infty}$.*

Proof. Proof of (a) and (d) follows directly from Proposition 5.5.12.

Since both $C(K)$ and X are constrained in $C(K, X)$, and the $\mathcal{B}$-IP ($IP_{f,\infty}$) is inherited by constrained subspaces, it follows that if $C(K, X)$ has the $\mathcal{B}$-IP ($IP_{f,\infty}$), then both $C(K)$ and X have the $\mathcal{B}$-IP ($IP_{f,\infty}$). Now by Theorem 5.4.4, if $C(K)$ has the $IP_{f,\infty}$, then K is extremally disconnected.

On the other hand, note that for every infinite K, c_0 is always contained in $C(K)$. Since c_0 lacks the $\mathcal{B}$-IP or the $IP_{f,\infty}$, (b) and necessity of (c) follows from Proposition 5.5.12.

For the 'sufficiency' part of (c), note that in case (i), $C(K, X)$ is a finite ℓ_∞-sum of X and in case (ii), $C(K, X)$ is a dual space. $\square$

In the next proposition, we prove a partial converse of Proposition 5.5.10(a) when Y is finite dimensional and K is compact and extremally disconnected.

Proposition 5.5.14. *Let $\mathcal{S}$ be any of the families $\mathcal{F}$, $\mathcal{K}$, $\mathcal{B}$ or $\mathcal{P}$. Let Y be a finite dimensional a $\mathcal{S}(Y)$-C-subspace of a Banach space X. Then for any extremally disconnected compact space K, $C(K, Y)$ is a $\mathcal{S}(C(K, Y))$-C-subspace of $C(K, X)$.*

Proof. We argue similar to the proof of [10, Proposition 4.11]. Let K be homeomorphically embedded in the Stone-Čech compactification $\beta(\Gamma)$ of a discrete set Γ and let $\phi : \beta(\Gamma) \to K$ be a continuous retract. Let $A \in \mathcal{S}(C(K, Y))$ and $g \in C(K, X)$. Note that since Y is finite dimensional, by the defining property of $\beta(\Gamma)$, any Y-valued bounded function on Γ has a norm preserving extension in $C(\beta(\Gamma), Y)$. Thus $C(\beta(\Gamma), Y)$ can be identified with $\bigoplus_{\ell_\infty(\Gamma)} Y$. Lift A to this space. In view of Theorem 5.5.6, this space is $\mathcal{S}(Y)$-C-subspace of $\oplus_\infty X$. This latter space contains $C(\beta(\Gamma), X)$. Thus by composing the functions with ϕ, we get a $f \in C(K, Y)$ such that $\|f - h\| \leq \|g - h\|$ for all $h \in A$. Hence the result. $\square$

And now for a partial converse of Proposition 5.5.10(b).

Theorem 5.5.15. *Let Y be a separable subspace of X. If Y is a AC-subspace of X, then for any standard Borel space (Ω, Σ) and any σ-finite measure μ, $L_p(\mu, Y)$ is a AC-subspace of $L_p(\mu, X)$.*

Proof. Let $f \in L_p(\mu, X)$. Since Y is a AC-subspace of X, for each $x \in X$, $\cap_{y \in Y} B_Y[y, \|x - y\|] \neq \emptyset$.

Define a multi-valued map $F : \Omega \longrightarrow Y$, by

$$F(t) = \begin{cases} \bigcap_{y \in Y} B_Y[y, \|f(t) - y\|] & \text{if} \quad f(t) \in X \setminus Y \\ \{f(t)\} & \text{otherwise} \end{cases}$$

Let $G = \{(t, z) : z \in F(t)\}$ be the graph of F.

Claim : G is a measurable subset of $\Omega \times Y$.

To establish the claim, we show that G^c is measurable. Since Y is separable, let $\{y_n\}$ be a countable dense set in Y. Observe that $z \notin F(t)$ if and only if either $f(t) \in Y$ and $z \neq f(t)$ or $f(t) \in X \setminus Y$ and there exists y_n such that $\|z - y_n\| > \|f(t) - y_n\|$. And hence,

$$\begin{aligned} G^c &= \Big\{(t, z) \in \Omega \times Y : f(t) \in Y \text{ and } z \neq f(t)\Big\} \cup \\ & \bigcup_{n \geq 1} \Big\{(t, z) \in \Omega \times Y : f(t) \in X \setminus Y \text{ and } \|z - y_n\| > \|f(t) - y_n\|\Big\} \end{aligned}$$

is a measurable set.

By von Neumann selection theorem (see [49, Theorem 7.1]), there is a measurable function $g : \Omega \longrightarrow Y$ such that $(t, g(t)) \in G$ for almost all $t \in \Omega$.

Observe that $\|g(t)\| \leq \|f(t)\|$ for almost all t. Hence $g \in L_p(\mu, Y)$. Also for any $h \in L_p(\mu, Y)$ we have $\|g(t) - h(t)\| \leq \|f(t) - h(t)\|$ for almost all t. Thus, $\|g - h\|_p \leq \|f - h\|_p$ for all $h \in L_p(\mu, Y)$. □

Question 5.1. *Suppose Y is a separable $\mathcal{K}$-C-subspace of X. Let (Ω, Σ, μ) be a probability space. Is $L^p(\mu, Y)$ a $\mathcal{K}$-C-subspace of $L^p(\mu, X)$?*

Remark 5.5.16. This question was answered in positive for $\mathcal{F}$-C-subspaces in [10] and for AC-subspaces above. Both the proofs are applications of von Neumann selection theorem. The problem here is that for a compact set $A \subseteq L^p(\mu, Y)$ and $\omega \in \Omega$, the set $\{f(\omega) : f \in A\}$ need not be compact in Y.

Chapter 6

Farthest points and nearest points

In this chapter, we study Mazur like intersection properties and its relation to farthest points. We also investigate the differentiability properties of the farthest distance and the distance maps.

6.1 Definition and preliminaries

Definition 6.1.1. For a closed and bounded set K in a Banach space X, the farthest distance map r_K is defined as $r_K(x) = \sup\{\|z - x\| : z \in K\}$, $x \in X$. r_K is a 1-Lipschitz continuous convex function.

For $x \in X$, we denote the set of points of K farthest from x by $Q_K(x)$, *i.e.*, $Q_K(x) = \{z \in K : \|z - x\| = r_K(x)\}$. Note that this set may be empty.

Let $D(K) = \{x \in X : Q_K(x) \neq \emptyset\}$.

The set of farthest points of K will be denoted by $\mathrm{far}(K)$, *i.e.*, $\mathrm{far}(K) = \cup\{Q_K(x) : x \in D(K)\}$.

K is densely remotal if $D(K)$ is norm dense in X.

For $x \in X$ and $\alpha > 0$, a crescent of K determined by x and α is the set $C(K, x, \alpha) = \{z \in K : \|z - x\| > r_K(x) - \alpha\}$.

K has Property (R) if any crescent of K contains a farthest point of K.

A point $k \in \mathrm{far}(K)$ is called a strongly farthest point if there is $x \in D(K)$ such that k is contained in crescents of K determined by x of arbitrarily small diameter.

Let us denote by $D_1(K)$ the set of points in X from which there are strongly farthest points in K.

A sequence $\{z_n\} \subseteq K$ is called a maximizing sequence for x if $\|x - z_n\| \longrightarrow r_K(x)$.

Note that if X is strictly convex (resp. LUR), then any farthest point of a closed bounded $K \subseteq X$ is an extreme (resp. denting) point of K.

Analogously, we define

Definition 6.1.2. For a closed set K in a Banach space X, we have already defined the distance function d_K and the metric projection P_K (see Definition 1.2.3). Note that d_K is a 1-Lipschitz function on X. However, d_K may not be convex. Also, for some $x \in X$, $P_K(x)$ may be empty.

Let $E(K) = \{x \in X \setminus K : P_K(x) \neq \emptyset\}$.

Again, we have already defined when K is called proximinal or Chebyshev. K is called almost proximinal if $E(K)$ is dense in $X \setminus K$.

For $x \in X$ and $\varepsilon > 0$, a metric slice of K determined by x and ε is the set $P_K(x, \varepsilon) = \{z \in K : \|z - x\| < r_K(x) + \varepsilon\}$.

A point $k \in K$ is called a strongly nearest point if there is $x \in E(K)$ such that k is contained in metric slices of K determined by x of arbitrarily small diameter.

Let us denote by $E_1(K)$ the set of points in $X \setminus K$ from which there are strongly nearest points in K.

A sequence $\{z_n\} \subseteq K$ is called a minimizing sequence for x if $\|x - z_n\| \longrightarrow d_K(x)$.

Definition 6.1.3. Let $h : X \to \mathbb{R}$ be a Lipschitz function. For $x, y \in X$ we define

$$h^0(x, y) = \limsup_{z \to x,\, t \to 0+} \frac{h(z + ty) - h(z)}{t}$$

and the generalized subdifferential (or the Clarke's subdifferential) of h at $x \in X$ is defined as

$$\partial^0 h(x) = \{x^* \in X^* : x^*(y) \leq h^0(x, y) \text{ for all } y \in X\}.$$

In general, $\partial^0 h(x)$ is a nonempty, w*-compact convex set in X^*. If h is Gâteaux differentiable at $x \in X$ with derivative $dh(x)$, then $dh(x) \in \partial^0 h(x)$. Also, $\partial^0 h$ is upper-semi-continuous in the sense that if $x_n \to x$ in norm, $f_n \in \partial^0 h(x_n)$ and f is a weak*-cluster point of $\{f_n\}$, then $f \in \partial^0 h(x)$.

Definition 6.1.4. A closed bounded convex set $K \subseteq X$ is said to be admissible if it is the intersection of closed balls containing it. Thus, X has the MIP if every closed bounded convex set in X is admissible.

A dual Banach space X^* is said to have the w*-MIP if every w*-compact convex set in X^* is admissible.

6.2 Remotely generated property

Let F be a norming subspace of X^*. Let us denote by σ the $\sigma(X, F)$ topology on X. Simplifying the assumptions in [2, 4] and correcting some minor inaccuracies of the assumptions in [18] and borrowing its terminology, we define :

Definition 6.2.1. A family $\mathcal{A}$ of σ-closed, norm bounded, convex sets is called a compatible family if

(A1) $\mathcal{A}$ is closed under arbitrary intersection, translation and scalar multiplication.

(A2) $C \in \mathcal{A}$, $A \subseteq C$ σ-closed convex $\Longrightarrow A \in \mathcal{A}$.

(A3) $C_1, C_2 \in \mathcal{A} \Longrightarrow \overline{\text{co}}^\sigma(C_1 \cup C_2) \in \mathcal{A}$.

Note that (A1) implies that $\mathcal{A}$ contains all singletons.

Example 6.2.2. Let F be any norming subspace of X^*.

(i) $\mathcal{A} = \{$all σ-closed bounded convex sets in $X\}$. This happens if and only if $B(X) \in \mathcal{A}$ and covers $\mathcal{B} = \{$all closed bounded convex sets in $X\}$, and if X is a dual space, $\mathcal{B}^* = \{$all w*-compact convex sets in $X\}$.

The property that every $K \in \mathcal{A}$ is admissible was called the F-MIP and characterized in [2] extending the characterization of the MIP and the w*-MIP [30].

(ii) $\mathcal{K} = \{$all compact convex sets in $X\}$. The property that every $K \in \mathcal{K}$ is admissible was called the (CI) and characterized in [67, 59].

(iii) $\mathcal{F} = \{$all compact convex sets in X with finite affine dimension$\}$. The property that every $K \in \mathcal{F}$ is admissible was characterized in [60].

(iii) $\mathcal{W} = \{$all weakly compact convex set in $X\}$. In [68], the authors obtained a sufficient condition for every $K \in \mathcal{W}$ to be admissible. A weaker sufficient condition was obtained in [2].

We now present the main theorem of this section. Some special cases of this were first observed in [4].

Theorem 6.2.3. *Let X be a strictly convex Banach space, F be a norming subspace of X^* and $\mathcal{A}$ be a compatible family. Then the following are equivalent :*

(a) *Every $K \in \mathcal{A}$ is admissible and has Property (R).*

(b) *Every $K \in \mathcal{A}$ is remotely σ-generated.*

Moreover, if $\mathcal{A}$ is as in Example 6.2.2 (i) above, then the assumption of strict convexity can be dropped.

Proof. $(a) \Rightarrow (b)$. Let $K \in \mathcal{A}$. Let $L = \overline{\text{co}}^{\sigma}(\text{far(K)})$. By $(A2)$, $L \in \mathcal{A}$. Suppose $K \setminus L \neq \emptyset$. By (a), L is admissible and hence, there exists a crescent C of K disjoint from L. Since K has the Property (R), $C \cap \text{far}(K) \neq \emptyset$. But, of course, $\text{far}(K) \subseteq L$.

$(b) \Rightarrow (a)$. Let $K \in \mathcal{A}$. If K lacks Property (R), there exists a crescent of K that is disjoint from $\text{far}(K)$, then it is disjoint from $\overline{\text{co}}^{\sigma}(\text{far}(K))$ as well. Hence $\overline{\text{co}}^{\sigma}(\text{far}(K)) \neq K$.

Now let X be strictly convex. Suppose there exists $K \in \mathcal{A}$ that is not admissible. Let

$$\widetilde{K} = \cap\{B : B \text{ closed ball and } K \subseteq B\}$$

Let $x_0 \in \widetilde{K} \setminus K$. Choose $y_0 \in K$ and $0 < \lambda < 1$ such that $z_0 = \lambda x_0 + (1-\lambda)y_0 \notin K$.

Let $K_1 = \mathrm{co}(K \cup \{z_0\})$. Then $K_1 \in \mathcal{A}$. We will show that $\mathrm{far}(K_1) \subseteq K$, and hence, K_1 is not remotely σ-generated.

Let $x \in X$. Then $\widetilde{K} \subseteq \{u \in X : \|u - x\| \leq r_K(x)\}$. Note that $r_K(x) \leq r_{K_1}(x) \leq r_{\widetilde{K}}(x) = r_K(x)$. Since z_0 as well as any point of the form

$$v = \alpha z_0 + (1-\alpha)z, \quad \alpha \in (0,1], \quad z \in K, \tag{6.1}$$

are not extreme points of $\widetilde{K}$ and X is strictly convex, they are not farthest points of $\widetilde{K}$ as well. Therefore, $\|v - x\| < r_K(x)$. Thus, $Q_{K_1}(x) \subseteq K$. Since $x \in X$ was arbitrary, $\mathrm{far}(K_1) \subseteq K$.

Observe that $K = \cap_{\lambda>0} K_\lambda$, where $K_\lambda = \overline{K + \lambda X_1}^{\sigma}$. And if $\mathcal{A} = \{$all σ-closed bounded convex sets in $X\}$, then $K_\lambda \in \mathcal{A}$. So passing to a K_λ, if necessary, we may assume that K has nonempty interior. Now if we choose $y_0 \in \mathrm{int}(K)$, then z_0 and any point of the form (6.1) are interior points of $\widetilde{K}$, and hence the result follows as before. □

We now obtain some sufficient conditions for Property (R).

Lemma 6.2.4. *Let $K \subseteq X$ be a bounded set. Let $x \in X$ and $\alpha > 0$ be given. Then there exists $\varepsilon > 0$ such that for any $y \in X$ with $\|x - y\| < \varepsilon$, there exists $\beta > 0$ such that $C(K, y, \beta) \subseteq C(K, x, \alpha)$.*

Proof. Take $0 < \varepsilon < \alpha/2$ and $0 < \beta < \alpha - 2\varepsilon$. □

Proposition 6.2.5. *Any densely remotal set has Property (R).*

Proof. If $D(K)$ is dense in X and $C(K, x, \alpha)$ is any crescent of K, then, by Lemma 6.2.4, there exists $y \in D(K)$ and $\beta > 0$ such that $C(K, y, \beta) \subseteq C(K, x, \alpha)$. Clearly, $Q_K(y) \subseteq C(K, y, \beta)$. Thus, K has Property (R). □

Theorem 6.2.6. (a) *If X has the RNP, then X^* has the w^*-MIP if and only if every w^*-compact convex set in X^* is w^*-remotely generated.*

(b) [43] *If X is reflexive, then X has the MIP if and only if every closed bounded convex set in X is remotely generated.*

(*c*) *If X is strictly convex and $\mathcal{A}$ is a compatible family of densely remotal sets—in particular, if $\mathcal{A}$ is one of $\mathcal{F}$, $\mathcal{K}$ or $\mathcal{W}$ of Example 6.2.2—then every $K \in \mathcal{A}$ is admissible if and only if every $K \in \mathcal{A}$ is remotely generated.*

Proof. If $K \subseteq X$ is weakly compact, by [43, Theorem 2.3], $D(K)$ contains a dense G_δ. And if X has the RNP, by [21, Proposition 3], the same conclusion holds for a w*-compact set K in X^*. Hence the result follows. $\square$

Remark 6.2.7. (a) and (b) above were also observed in [4]. (c) gives the only known characterization of when every $K \in \mathcal{W}$ is admissible.

Question 6.1. *Can one replace Theorem 6.2.3(a) by the apparently stronger condition that every $K \in \mathcal{C}$ is admissible and densely remotal? Or, by the apparently weaker condition that every $K \in \mathcal{C}$ is admissible and has farthest points?*

We now recall the following example from [4] that shows that the MIP alone is not sufficient to ensure that every closed bounded convex set in X is remotely generated. This also gives an example that if X^* has RNP, there may exist a closed bounded convex set in X with $\mathrm{far}(K) = \emptyset$ (Compare this with [21, Proposition 3]).

Example 6.2.8. The space c_0 has an equivalent strictly convex Fréchet differentiable norm (see e.g., [20, Theorem II.7.1]), and hence, has the MIP. However, since the unit ball of the sup norm on c_0 lacks extreme points, it must lack farthest points in the new norm.

Observe that since c_0 is Asplund, and X has the MIP implies X^{**} has the w*-MIP, Theorem 6.2.6(a) shows that when equipped with the bidual of the above norm, every w*-compact convex set in ℓ^∞ is remotely w*-generated. Thus, there is closed bounded convex set $K \subseteq c_0$, such that no farthest point of the w*-closure of K in X^{**} comes from K.

Question 6.2. *If every closed bounded convex set in X is remotely generated, does it follow that every w*-compact convex set in X^{**} is w*-remotely generated?*

As noted above, since a separable MIP space is Asplund, the answer is yes for separable spaces. On the other hand, from [18], it follows that if X has the MIP, then every w*-compact convex set in X^{**} is intersection of balls centred at points of X. Thus, one only needs to verify that every crescent of a w*-compact convex set K determined by some $x \in X$ contains a farthest point of K.

Question 6.3. *If every closed bounded convex set in X have Property (R), is X necessarily reflexive?*

6.3 Strongly farthest and strongly nearest points

In the following proposition(s), we collect some properties of strongly farthest (nearest) points and the set $D_1(K)$ $(E_1(K))$.

Proposition 6.3.1. *Let K be closed bounded set in a Banach space X.*

- (a) *$k \in K$ is strongly farthest from $x \in X$ if and only if any maximizing sequence for x converges to k. In fact, it suffices if any maximizing sequence for x converges.*
- (b) *A strongly farthest point of K is a strongly exposed point of K.*
- (c) *If $x \in D_1(K)$, then Q_K is single-valued and continuous at x.*
- (d) *$D_1(K)$ is a G_δ in X.*

Proof. (a). Let $k \in K$ be strongly farthest from $x \in X$. Suppose $\{z_n\} \subseteq K$ is a maximizing sequence for x. Let $C = C(K, x, \alpha)$ be a crescent of K. Then, for all sufficiently large n, $z_n \in C$. Therefore, $\|z_n - k\| \le \operatorname{diam}(C)$. Since k is contained in crescents of arbitrarily small diameter, $z_n \to k$.

Conversely, if $k \in K$ is not strongly farthest from x, then there exists $\varepsilon > 0$ and $z_n \in C(K, x, 1/n)$ such that $\|z_n - k\| \ge \varepsilon$. Then $\{z_n\}$ is a maximizing sequence that does not converge to k.

Now if there are two maximizing sequences $\{u_n\}$ and $\{v_n\}$ converging to different limits, then by taking $w_{2n} = u_n$ and $w_{2n+1} = v_n$, we get a maximizing sequence $\{w_n\}$ that does not converge at all.

(b). Let $k \in K$ be strongly farthest from $x \in X$. Let $x^* \in X^*$, $\|x^*\| = 1$ be such that $x^*(k - x) = \|k - x\| = r_K(x)$. We will show that x^* strongly exposes $k \in K$.

Since for any $z \in K$, $x^*(z - x) \le \|z - x\| \le r_K(x)$, $\sup x^*(K) = r_K(x) + x^*(x) = x^*(k)$. It follows that if $\{z_n\} \subseteq K$ is such that $x^*(z_n) \to \sup x^*(K)$, then $\{z_n\}$ is a maximizing sequence, and therefore, converges to k.

(c). If $x \in D_1(K)$, then Q_K is clearly single-valued. Let $Q_K(x) = \{k\}$. Now suppose $x_n \to x$ and $z_n \in Q_K(x_n)$, then, $\{z_n\}$ is a maximizing sequence for x, and hence, $z_n \to k$.

(d). For $n \ge 1$, let

$$G_n = \{x \in X : \text{ there exists } \alpha > 0 \text{ such that } \operatorname{diam}(C(K, x, \alpha)) < 1/n\}.$$

Then, by Lemma 6.2.4, G_n is open and by the Cantor Intersection Theorem, $D_1(K) = \cap_n G_n$. □

Analogous results for nearest points can be proved essentially similarly, once we observe that

Lemma 6.3.2. *Let $K \subseteq X$ be a closed set. Let $x \in X$ and $\delta > 0$ be given. Then there exists $\varepsilon > 0$ such that for any $y \in X$ with $\|x - y\| < \varepsilon$, there exists $\eta > 0$ such that $P_K(y, \eta) \subseteq P_K(x, \delta)$.*

Proof. Just as in Lemma 6.2.4, take $0 < \varepsilon < \delta/2$ and $0 < \eta < \delta - 2\varepsilon$. □

Proposition 6.3.3. *Suppose K is a closed set in X.*

- (a) *$k \in K$ is a strongly nearest point from $x \in X \setminus K$ if and only if any minimizing sequence for x converges to k.*
- (b) *If $x \in E_1(K)$ then the metric projection P_K is single valued and continuous at x.*
- (c) *$E_1(K)$ is a G_δ in $X \setminus K$.*

We now give a natural setting for existence of strongly farthest points.

Theorem 6.3.4. *A Banach space X is strictly convex if and only if for every compact set $K \subseteq X$, every farthest point of K is strongly farthest.*

Proof. Let X be strictly convex and $K \subseteq X$ be compact. Let $x \in X$ and $k \in Q_K(x)$. Let $t > 1$ and $y = k + t(x - k)$. Note that for any $z \in K$,

$$\|z-y\| = \|z-k-t(x-k)\| = \|z-x+(1-t)(x-k)\| \leq \|z-x\|+(t-1)\|(x-k)\|$$

Therefore, $r_K(y) = t\, r_K(x)$ and $k \in Q_K(y)$. Moreover, if $k_1 \in Q_K(y)$, then for $z = k_1$, equality holds everywhere and hence, $k_1 \in Q_K(x)$. Further, by strict convexity, $k_1 - x$ is a scalar multiple of $k - x$, and hence, $k = k_1$. That is, $Q_K(y) = \{k\}$.

Let $\{k_n\} \subseteq K$ be a maximizing sequence for y. Since K is compact, $\{k_n\}$ has a subsequence converging to some $k_0 \in K$. Then $k_0 \in Q_K(y)$ and, by the above, $k_0 = k$. That is, every maximizing sequence for y has a subsequence converging to k. It follows that every maximizing sequence for y converges to k, and hence, k is a strongly farthest point from y.

Conversely, if X is not strictly convex, there exist unit vectors x, y such that the line segment $[x, y]$ is contained in the unit sphere. Clearly, $K = [x, y]$ is compact and any point of K is farthest from 0. But no point of K except the end-points is extreme and therefore, such points cannot be strongly farthest. □

Combining this with Theorem 6.2.6(c), we get

Corollary 6.3.5. *Let X be a strictly convex Banach space. Then every compact convex set in X is admissible if and only if every compact convex set in X is the closed convex hull of its strongly farthest points.*

Similar results for a general closed, bounded set would naturally require stronger convexity assumptions on the norm.

Theorem 6.3.6. *A Banach space X is LUR if and only if for every closed and bounded set $K \subseteq X$, every farthest point of K is strongly farthest.*

Proof. Suppose X is LUR. Observe that every unit vector is a farthest point of X_1 from 0. For any unit vector x and any sequence $\{x_n\} \subseteq X_1$ that is maximizing for $-x$, we have $\|x + x_n\| \to 2$. By LUR, $x_n \to x$. Thus x is a strongly farthest point from $-x$ in X_1.

Now let K be a closed and bounded set in X, $x \in X$ and $k \in Q_K(x)$. Then $K \subseteq B[x, r_K(x)]$ and $\|k - x\| = r_k(x)$. By a suitable translation and scaling, the above argument shows that k is a strongly farthest point of $B[x, r_K(x)]$ from $y = 2x - k$. Since $K \subseteq B[x, r_K(x)]$, it is clear that $r_K(y) = 2r_K(x)$ and k is strongly farthest point of K from y.

Conversely, suppose in particular that every unit vector is a strongly farthest point of X_1. It follows that every unit vector is a strongly exposed point of X_1, and therefore, X is strictly convex.

Now let $x_0 \in X_1$, $\|x_0\| = 1$, be strongly farthest from some $x \in X$. Then $\|x - x_0\| = r_{B(X)}(x) = 1 + \|x\|$. By strict convexity, it follows that $x = \alpha x_0$ for some $\alpha \in \mathbb{R}$ and $|\alpha - 1| = 1 + |\alpha|$. Therefore, $\alpha < 0$.

To show X is LUR, let $\{x_n\} \subseteq X_1$ be such that $\|x_n + x_0\| \to 2$. For each n consider the function on $(0, 1)$,

$$f_n(\lambda) = 1 - \|\lambda x_n + (1 - \lambda)x_0\|$$

Then for all $\lambda \in (0, 1)$, $f_n(\lambda) \geq 0$. And by triangle inequality,

$$2f_n(1/2) \geq f_n(\lambda) + f_n(1 - \lambda) \geq f_n(\lambda) \geq 0$$

By assumption, $f_n(1/2) \to 0$. Thus, for any $\lambda \in (0, 1)$, $f_n(\lambda) \to 0$. In particular, putting $\lambda = 1/(1-\alpha)$, we get $\|x_n - \alpha x_0\| \to (1-\alpha)$, that is $\{x_n\}$ is a maximizing sequence for $x = \alpha x_0$. Hence, $x_n \to x_0$. □

Corollary 6.3.7. *Let X be a reflexive LUR Banach space. Then X has the MIP if and only if every closed bounded convex set in X is the closed convex hull of its strongly farthest points.*

Question 6.4. *If X is a LUR Banach space with the MIP, does every closed bounded convex set in X with the RNP have Property (R)?*

The example in [21, Proposition 1] shows that RNP alone is not enough to ensure even the existence of farthest points. But then, the set under consideration there is not admissible.

It is easy to see that if X has the MIP and $K \subseteq X$ is a closed and bounded set with RNP, then any crescent of K contains a crescent of K of arbitrarily small diameter. If with the additional assumption of LUR, one could show that the set G_n defined in the proof of Proposition 6.3.1(d) is dense, it would follow that K is densely strongly remotal.

We can, however, show that in a LUR Banach space, any densely remotal set is densely strongly remotal.

Proposition 6.3.8. *Suppose X is LUR and $K \subseteq X$ is densely remotal. Then $D_1(K)$ is a dense G_δ in X.*

Proof. By Proposition 6.3.1(d), it suffices to show that $D_1(K)$ is dense in X.

Let $x \in D(K)$. Let $k \in Q_K(x)$. $0 < \varepsilon < 1$. Let $y = k + (1+\varepsilon)(x-k)$. Then, $\|x - y\| = \varepsilon r_K(x)$. As in the proof of Theorem 6.3.4, $r_K(y) = (1+\varepsilon)r_K(x)$ and by strict convexity, k is a unique farthest point from y.

We now claim k is a strongly farthest point from y. Let $\{z_n\} \subseteq K$ be a maximizing sequence for y. That is, $\|z_n - y\| \to (1+\varepsilon)r_K(x)$. Then,

$$\left\| \frac{(z_n - x) + \varepsilon(k - x)}{(1+\varepsilon)} \right\| \to r_K(x)$$

Then $y_n = (z_n - x)/r_K(x) \in X_1$ and $y_0 = (k - x)/r_K(x)$, $\|y_0\| = 1$, and for $\lambda = 1/(1+\varepsilon)$, we have $\|\lambda y_n + (1-\lambda)y_0\| \to 1$. Notice that since $\varepsilon < 1$, $1/2 < \lambda < 1$.

As in the proof of Theorem 6.3.6, let

$$f_n(\lambda) = 1 - \|\lambda y_n + (1-\lambda)y_0\|.$$

Since

$$\lambda y_n + (1-\lambda)y_0 = (2-2\lambda)\frac{y_n + y_0}{2} + (2\lambda - 1)y_n$$

using convexity of the norm, we get that

$$\|\lambda y_n + (1-\lambda)y_0)\| \leq (2-2\lambda)\left\|\frac{y_n + y_0}{2}\right\| + (2\lambda - 1)$$

It follows that

$$f_n(\lambda) \geq (2-2\lambda)f_n(1/2) \geq 0.$$

Since $f_n(\lambda) \to 0$, we have that $f_n(1/2) \to 0$, that is, $\|y_n + y_0\| \to 2$. Since X is LUR, $y_n \to y_0$ and hence, $z_n \to k$. $\square$

Corollary 6.3.9. (*a*) *For any weakly compact set K in a LUR Banach space, $D_1(K)$ is residual, i.e., contains a dense G_δ in X.*

(*b*) *[26, Corollary 2.8] If X^* is Fréchet smooth, then for any closed and bounded subset $K \subseteq X$, $D_1(K)$ is residual.*

Analogously we have,

Theorem 6.3.10. *Let X be a LUR Banach space and K be a closed almost proximinal set in X. Then $E_1(K)$ is a dense G_δ in $X \setminus K$.*

Proof. As before, it suffices to show that $E_1(K)$ is dense in $E(K)$. So, let $x \in E(K)$ and $k_0 \in P_K(x)$. We show that given any $0 < \varepsilon < 1/3$, the point $x_0 = x - \varepsilon(x - k_0) \in E_1(K)$.

We note that $d_K(x_0) = (1-\varepsilon)d_K(x)$. Let $\{k_n\}$ be a minimizing sequence for x_0. It is easy to observe that $\|x-k_n\| \to d_K(x)$ as well. Now, $\|x_0-k_n\| \to d_K(x_0)$, that is,

$$\|k_n - x + \varepsilon(x - k_0)\| \to (1-\varepsilon)d_K(x).$$

Since

$$\begin{aligned} & \frac{1}{d_K(x)}\|k_n - x - \varepsilon(k_0 - x)\| + \|k_n - x\|\left[\frac{1}{d_K(x)} - \frac{1}{\|x - k_n\|}\right] \\ \geq & \left\|\frac{k_n - x}{\|x - k_n\|} - \varepsilon\frac{k_0 - x}{d_K(x)}\right\| \geq 1 - \varepsilon, \end{aligned}$$

we have

$$\left\|\frac{k_n - x}{\|x - k_n\|} - \varepsilon\frac{k_0 - x}{d_K(x)}\right\| \to 1 - \varepsilon$$

Let $u_n = (k_n - x)/\|x - k_n\|$, $u_0 = (k_0 - x)/d_K(x)$ and $\lambda = (1-2\varepsilon)/(1-\varepsilon)$. Note that, since $e < 1/3$, $1/2 < \lambda < 1$. Then $\|u_n\| = \|u_0\| = 1$ and

$$\|2u_n - [\lambda u_n + (1-\lambda)u_0]\| \to 1.$$

Since

$$\|2u_n - (\lambda u_n + (1-\lambda)u_0)\| \geq 2 - \|\lambda u_n + (1-\lambda)u_0\| \geq 1$$

we have $\|\lambda u_n + (1-\lambda)u_0)\| \to 1$ as well. As before, let

$$f_n(\lambda) = 1 - \|\lambda u_n + (1-\lambda)u_0)\|.$$

Arguing as before, since the norm is LUR, $u_n \to u_0$ and hence $k_n \to k_0$. $\square$

For the rest of this section, we concentrate on (sub-)differentiability of the farthest distance map r_K and the distance function d_K. Since r_K is a convex function, we can work with its subdifferential ∂r_K. For d_K, we need to work with its generalized subdifferential $\partial^0 d_K$.

Theorem 6.3.11. *Let $x \in D_1(K)$. Then, $\partial r_K(x) = \mathcal{D}(x-k)$, where $k \in Q_K(x)$. Moreover, r_K is Gâteaux (resp. Fréchet) differentiable at x if and only if the norm is Gâteaux (resp. Fréchet) differentiable at $x-k$.*

Proof. Let $x^* \in \mathcal{D}(x-k)$. Then $x^*(x-k) = r_K(x)$. For $z \in X$, $x^*(z-x) = x^*(z) - x^*(k) - r_K(x) \le r_K(z) - r_K(x)$. Thus $x^* \in \partial r_K(x)$.

Conversely, let $x^* \in \partial r_K(x)$. Since $\mathcal{D}(x-k)$ is a w*-closed convex subset of the unit sphere of X^*, it is enough to show that for any unit vector z, there is an $x_0^* \in \mathcal{D}(x-k)$ such that $x^*(z) \le x_0^*(z)$.

Let $\{k_n\} \subseteq K$ be such that $\|x + z/n - k_n\| > r_K(x + z/n) - 1/n^2$. Then $\{k_n\}$ is a maximizing sequence for x, and hence, $k_n \to k$. Now

$$x^*(\frac{z}{n}) = x^*(x+\frac{z}{n}) - x^*(x) \le r_K(x+\frac{z}{n}) - r_K(x) < \|x+\frac{z}{n}-k_n\| - r_K(x) + \frac{1}{n^2}.$$

Choose $x_n^* \in \mathcal{D}(x + z/n - k_n)$. Then

$$x_n^*(\frac{z}{n}) = x_n^*(x + \frac{z}{n} - k_n) - x_n^*(x-k_n) \ge \|x + \frac{z}{n} - k_n\| - r_K(x).$$

Combining the two, we have $x^*(z) \le x_n^*(z) + 1/n$. Let x_0^* be a w*-cluster point of $\{x_n^*\}$. Since $x + z/n - k_n$ converges to $x-k$ in norm, we have $x_0^* \in \mathcal{D}(x-k)$ and $x^*(z) \le x_0^*(z)$, as desired.

Thus, the norm is Gâteaux differentiable is at $x-k \Leftrightarrow \mathcal{D}(x-k)$ is singleton $\Leftrightarrow$ so is $\partial r_K(x) \Leftrightarrow r_K$ is Gâteaux differentiable at x.

Now, let $\{x^*\} = \partial r_K(x) = \mathcal{D}(x-k)$. For any $\lambda \in \mathbb{R}$ and $z \in B(X)$, $x^*(\lambda z) \le \|x+\lambda z-k\| - \|x-k\| \le r_K(x+\lambda z) - r_K(x)$. Therefore,

$$\left|\frac{\|x+\lambda z-k\| - \|x-k\|}{\lambda} - x^*(z)\right| \le \left|\frac{r_K(x+\lambda z) - r_K(x)}{\lambda} - x^*(z)\right|.$$

Thus, Fréchet differentiability of r_K at x implies that of the norm at $x-k$.

Conversely, let the norm be Fréchet differentiable at $x-k$. Let $x_n \to x$, $x_n^* \in \partial r_K(x_n)$ and $x^* \in \partial r_K(x)$, then $\{x_n^*\} \subseteq X_1^*$ and since r_K is Gâteaux differentiable at x, $x_n^* \to x^*$ in the w*-topology. Since $x^* \in \mathcal{D}(x-k)$, x^* is a w*-PC of X_1^*, and therefore, $x_n^* \to x^*$ in norm. It follows that r_K is Fréchet differentiable at x. □

Remark 6.3.12. [26, Theorem 3.2(a)] proves only the "necessity" part of this result. Our proof is also simpler.

Combining Theorem 6.3.11 with Theorem 6.3.8, it follows that in a Banach space with smooth LUR norm, the farthest distance map r_K of a densely remotal set K is Gâteaux differentiable on a dense G_δ.

The analogue of the above theorem for nearest points is :

Theorem 6.3.13. *Let $x \in E_1(K)$. Then*

$$\partial^0 d_K(x) \subseteq \mathcal{D}(x - P_K(x)).$$

The equality if the norm on X is smooth at $x - P_K(x)$. Moreover, if the norm on X is Fréchet smooth at $x - P_K(x)$, d_K is Fréchet smooth at x.

Proof. Let $f \in \partial^0 d_K(x)$. Since $\mathcal{D}(x-P_K(x))$ is a w*-compact convex set, it is enough to show that given any $y \in X$, $\|y\| = 1$, there is a $g \in \mathcal{D}(x-P_K(x))$ such that $f(y) \le g(y)$.

By definition of $\partial^0 d_K(x)$ given any $\varepsilon > 0$ there exist $z_n \to x$ and $t_n \to 0+$ such that

$$f(y) - \varepsilon \le \frac{d_K(z_n + t_n y) - d_K(z_n)}{t_n}.$$

Get $k_n \in K$ such that $\|z_n - k_n\| < d_K(z_n) + t_n^2$. Then k_n is minimizing for x as well and hence $k_n \to P_K(x)$. Now

$$f(y) - \varepsilon < \frac{\|z_n + t_n y - k_n\| - \|z_n - k_n\| + t_n^2}{t_n}.$$

By mean value property of subdifferential (see [19, Theorem 2.3.7]), there exist $g_n \in \mathcal{D}(w_n)$ such that $g_n(t_n y) = \|z_n + t_n y - k_n\| - \|z_n - k_n\|$, where w_n lies on the line $[z_n - k_n, z_n + t_n y - k_n]$. Thus $f(y) - \varepsilon < g_n(y) + t_n$. Let g be a w*-cluster point of $\{g_n\}$. Since $w_n \to x - P_K(x)$ in norm, by the upper semicontinuity of $\mathcal{D}$, we have $g \in \mathcal{D}(x-k)$ and $f(y) - \varepsilon \leq g(y)$. Thus we have the desired result.

If the norm on X is smooth at $x - P_K(x)$, $\mathcal{D}(x - P_K(x))$ is singleton and so is $\partial^0 d_K(x)$. Note that this implies d_K is smooth at x.

If the norm is Fréchet smooth at $x - P_K(x)$, note that for $f \in \partial^0 d_K(x)$ and any $z \in X$, $\|z\| = 1$, we have,

$$f(z) = \lim \frac{d_K(x + \frac{1}{n}z) - d_K(x)}{\frac{1}{n}} \leq \lim \frac{\|x - P_K(x) + \frac{1}{n}z\| - \|x - P_K(x)\|}{\frac{1}{n}}$$

the right hand side converges to $f(z)$ uniformly over all $\|z\| = 1$. Thus d_K is Fréchet smooth at x. $\square$

The following corollary is clear from Theorem 6.3.13 and Proposition 6.3.3. In [14, Theorem 10] the authors showed this for Hilbert spaces.

Corollary 6.3.14. *Let X be LUR and (Fréchet) smooth. Then for an almost proximinal set K, d_K is (Fréchet) smooth on a residual subset of $X \setminus K$.*

We now obtain an expression for ∂r_K in the spirit of Preiss' Theorem [52]. Note that our result does not need smoothness of the norm and with (Fréchet) smoothness, by Theorem 6.3.11, we get back Preiss' Theorem for ∂r_K. A similar result is obtained in [66, Proposition 4.3] under significantly stronger assumptions.

Theorem 6.3.15. *Let K be such that $D_1(K)$ is dense in X and $x \in X$. Then*

$$\partial r_K(x) = \bigcap_{\delta > 0} \overline{co}^{w^*}\{\partial r_k(y) : y \in D_1(K) \text{ and } \|y - x\| < \delta\}.$$

Proof. Let $x^* \in RHS$ and $\varepsilon > 0$. Choose $\delta < \varepsilon/3$. For $z \in X$, choose $y \in D_1(K)$ and $y^* \in \partial r_K(y)$ such that $\|y-x\| < \delta$ and $x^*(z-x) < y^*(z-x)+\delta$. Thus,

$$\begin{aligned} x^*(z-x) &< y^*(z-x)+\delta = y^*(z-y)+y^*(y-x)+\delta \\ &\leq r_K(z)-r_K(y)+2\delta \leq r_K(z)-r_K(x)+3\delta \\ &\leq r_K(z)-r_K(x)+\varepsilon. \end{aligned}$$

Since ε is arbitrary, we have $x^* \in \partial r_K(x)$.

Conversely, let $x^* \in \partial r_K(x)$. As in Theorem 6.3.11, we will show given any unit vector z there is an $x_0^* \in RHS$ such that $x^*(z) \leq x_0^*(z)$.

For each n, get $y_n \in D_1(K)$ such that $\|x+z/n-y_n\| < 1/n^2$. Then

$$x^*(\frac{z}{n}) \leq r_K(x+\frac{z}{n}) - r_K(x) \leq r_K(y_n) - r_K(y_n - \frac{z}{n}) + \frac{2}{n^2}.$$

Let $x_n^* \in \partial r_K(y_n)$. Let $k_n \in Q_K(y_n)$. Then

$$x_n^*(\frac{z}{n}) = x_n^*(y_n-k_n) - x_n^*(y_n-\frac{z}{n}-k_n) \geq r_K(y_n) - r_K(y_n-\frac{z}{n}).$$

Thus $x^*(z) \leq x_n^*(z)+2/n$. Let x_0^* be a w*-cluster point of x_n^*. Then $x_0^* \in RHS$ and $x^*(z) \leq x_0^*(z)$. $\square$

Similar exercise with $\partial^0 d_K$ yields a *proximal normal formula* in smooth LUR Banach spaces.

Let $E_1(K)$ be dense in $X \setminus K$. For $x \in X \setminus K$ denote by $D_x(K)$ the w*-cluster points of $\mathcal{D}(y_n - P_K(y_n))$ where $y_n \in E_1(K)$, $y_n \to x$.

Theorem 6.3.16. *Let X be smooth. Let K be a closed set in X such that $E_1(K)$ is dense in $X \setminus K$. Then for any $x \in X \setminus K$ we have,*

$$\partial^0 d_K(x) = \overline{co}^{w^*}\{D_x(K)\}.$$

Proof. Let $f \in D_x(K)$. Then there is a sequence $\{y_n\} \in E_1(K)$, $y_n \to x$, $f_n \in \mathcal{D}(y_n - P_K(y_n))$ such that $f_n \to f$ in w*-topology. By Theorem 6.3.13, $f_n \in \partial^0 d_K(y_n)$. By upper semicontinuity of $\partial^0 d_K$, we have $f \in \partial^0 d_K(x)$. Since $\partial^0 d_K(x)$ is weak*-closed convex set, $\overline{co}^{w^*}\{D_x(K)\} \subseteq \partial^0 d_K(x)$.

Conversely, let $f \in \partial^0 d_K(x)$. As before, it is enough to show that for any $y \in X$, $\|y\| = 1$, there exists $g \in \overline{co}^{w^*}\{D_x(K)\}$ such that $f(y) \leq g(y)$.

Given $\varepsilon > 0$ there are $z_n \in X \setminus K$, $z_n \to x$ and $t_n \to 0+$ such that for each n,

$$f(y) - \varepsilon \leq \frac{d_K(z_n + t_n y) - d_K(z_n)}{t_n}.$$

Choose $y_n \in E_1(K)$ such that $\|z_n + t_n y - y_n\| < t_n^2$. Then

$$d_K(z_n + t_n y) \leq d_K(y_n) + t_n^2$$

and

$$d_K(z_n) > d_K(y_n - t_n y) - t_n^2.$$

Thus,

$$f(y) - \varepsilon \leq \frac{d_K(y_n) - d_K(y_n - t_n y)}{t_n} + 2t_n \leq d_K^0(y_n, y) + 2t_n.$$

Since by Lemma 6.3.13, $\partial^0 d_K(y_n) = \mathcal{D}(y_n - P_K(y_n))$ is singleton we have $d_K^0(y_n, y) = g_n(y)$ where $g_n(y_n - P_K(y_n)) = \|y_n - P_K(y_n)\|$. Let g be a w*-cluster point of g_n. We then have $f(y) - \varepsilon \leq g(y)$ and the result follows. $\square$

Thus we have a *proximal normal formula* for X smooth and LUR.

Corollary 6.3.17. *Suppose K is a closed set in X such that $E_1(K)$ is dense in $X \setminus K$. Then for $x \in bdyK$, the normal cone $N_K(x)$ (see Definition 1.1.7) is the w*-closed convex cone generated by the origin and $D_x(K)$.*

6.4 Some applications

We first present a result on range of ∂r_K. Compare this with [66, Theorem 4.2]. We will need the following Lemma.

Lemma 6.4.1. *[66, Lemma 4.1] Let $\{x_n\}$ be a sequence in X such that $\lim \|x_n\| = \infty$ and let $x_n^* \in \partial r_K(x_n)$. Then, for any $y \in X$,*

$$\lim x_n^* \left(\frac{x_n - y}{\|x_n - y\|} \right) = \lim \|x_n^*\| = 1$$

Theorem 6.4.2. *Let X be a smooth (resp. Fréchet smooth) Banach space. Let $K \subseteq X$ be a closed and bounded set such that $D_1(K)$ is dense in X, then the image of $D_1(K)$ under ∂r_K is w*-dense (resp. norm dense) in the unit sphere of X^*.*

Proof. Let $NA(X)$ denote the set of norm attaining functionals on the unit sphere of X^*, which by Bishop-Phelps Theorem, is norm dense there. Let $x_0^* \in NA(X)$ and $x_0 \in X$, $\|x\| = 1$, be such that $x_0^*(x_0) = 1$. By density of $D_1(K)$, choose $x_n \in D_1(K)$ such that $\|x_n - nx_0\| < 1/n$ and let $x_n^* \in \partial r_K(x_n)$. Then $\|x_n\| \to \infty$. Therefore, by Lemma 6.4.1, $\lim x_n^*(x_n/\|x_n\|) = \lim \|x_n^*\| = 1$. But since $x_n/\|x_n\| \to x_0$ in norm, $x_n^*(x_0) \to 1$ as well. Thus, any w*-cluster point of $\{x_n^*\}$ is in $\mathcal{D}(x_0)$. Since the norm is smooth, this set is singleton. Hence, $x_n^* \to x_0^*$ in w*-topology.

Now, if the norm on X is Fréchet smooth, then x_0^* chosen above is a w*-PC of X_1^*. Thus $\partial r_K(D_1(K))$ is norm dense in the dual unit sphere. $\square$

We conclude this section with a result on continuity of metric projection on Chebyshev sets. Our result, in conjunction with a result of Vlasov, gives a necessary and sufficient condition for convexity of Chebyshev sets in a Banach space X such that both X and X^* are LUR. The classical result of Vlasov (see [64]) we refer to is the following :

Theorem 6.4.3. *If X^* is strictly convex, then every Chebyshev set in X with continuous metric projection is convex.*

We will also need the following lemma.

Lemma 6.4.4. *[14, Lemma 1] For any $x \in E(K)$ and every $k \in P_K(x)$, there exists an $f \in \mathcal{D}(x - k)$ such that $f \in \partial^0 d_K(x)$.*

Proposition 6.4.5. *Suppose X is both LUR and Fréchet smooth. Then for a Chebyshev set $K \subseteq X$ and $x \in X \setminus K$, the metric projection P_K is continuous at x if and only if $\partial^0 d_K(x)$ is singleton.*

Proof. Let $K \subseteq X$ be a Chebyshev set and $x \in X \setminus K$ be such that P_K is continuous at x. From Theorem 6.3.10 and Theorem 6.3.16, we have $\partial^0 d_K(x) = \overline{co}^{w^*}\{D_x(K)\}$.

Now, let $f \in D_x(K)$. By definition of $D_x(K)$, there exists $\{x_n\} \subseteq E_1(K)$, $x_n \to x$ and $f_n \in \mathcal{D}(x_n - P_K(x_n))$ such that $f_n \to f$ in w*-topology. But by continuity of P_K, $x_n - P_K(x_n) \to x - P_K(x)$. Therefore, $f \in \mathcal{D}(x - P_K(x))$. That is, $D_x(K) \subseteq \mathcal{D}(x - P_K(x))$. And hence, $\partial^0 d_K(x) = \overline{co}^{w^*}\{D_x(K)\} \subseteq \mathcal{D}(x - P_K(x))$ as well. Now, since X is smooth, $\partial^0 d_K(x)$ must be singleton.

Conversely, let $K \subseteq X$ be a Chebyshev set and $x \in X \setminus K$ be such that $\partial^0 d_K(x)$ singleton. By Lemma 6.4.4, this implies $\partial^0 d_K(x) \subseteq \mathcal{D}(x - P_K(x))$. Since X is smooth, we actually have $\partial^0 d_K(x) = \mathcal{D}(x - P_K(x))$.

Now, let $y_n \in X \setminus K$, $y_n \to x$. We want to show $P_K(y_n)$ converges to $P_K(x)$. Define $x_n = y_n - \frac{1}{n}(y_n - P_K(y_n))$. By the proof of Theorem 6.3.10, $x_n \in E_1(K)$ and $P_K(x_n) = P_K(y_n)$ for all $n > 3$. Note that $x_n \to x$ as well and by Theorem 6.3.13, $\partial^0 d_K(x_n) \subseteq \mathcal{D}(x_n - P_K(x_n))$. Let $f_n \in \partial^0 d_K(x_n)$ and let f be a w*-cluster point of f_n. Then by upper semicontinuity of $\partial^0 d_K$, $f \in \partial^0 d_K(x) = \mathcal{D}(x - P_K(x))$. Since X is Fréchet smooth, this would imply that $f_n \to f$ in norm as well. And hence,

$$f\left(\frac{x_n - P_K(x_n)}{\|x_n - P_K(x_n)\|}\right) \to 1$$

Now since the norm on X is LUR, f strongly exposes $\frac{x - P_K(x)}{\|x - P_K(x)\|}$. Thus $\frac{x_n - P_K(x_n)}{\|x_n - P_K(x_n)\|} \to \frac{x - P_K(x)}{\|x - P_K(x)\|}$. Since $\|x_n - P_K(x_n)\| = d_K(x_n) \to d_K(x) = \|x - P_K(x)\|$, we have $P_K(x_n) = P_K(y_n) \to P_K(x)$ as desired. $\square$

Theorem 6.4.6. *Suppose the norms on X and X^* are LUR. Then a Chebyshev set K is convex in X if and only if $\partial^0 d_K(x)$ is singleton for all $x \in X \setminus K$.*

Proof. If K is convex then $\partial^0 d_K$ coincides with usual subdifferential of d_K and if the norm on X^* is LUR then d_K is Fréchet smooth at each $x \in X \setminus K$ (see [20, page 365]). Thus $\partial^0 d_K(x)$ is singleton for each such x.

Conversely let, $\partial^0 d_K(x)$ is singleton for each $x \in X \setminus K$. By Proposition 6.4.5, we have the metric projection on K is continuous. Thus by Theorem 6.4.3, K is convex. $\square$

Remark 6.4.7. (a) d_K being a Lipschitz function, the condition $\partial^0 d_K(x)$ is singleton for all $x \in X \setminus K$ reduces to strict differentiability of d_K (see [19], page 30 for definition and Proposition 2.2.4 for the equivalence of these two). In particular, this is satisfied when d_K is continuously differentiable on $X \setminus K$.

(b) In [27, Theorem 3.6], the author showed that for a closed set K in a Banach space X with the norms of X and X^* are Fréchet differentiable, if for each $x \in X \setminus K$ there exists a unit vector $u \in X$ such that the directional derivative $D_u d_K(x) = 1$ then K is convex. A close look at the proof given in that paper actually shows that this condition implies $E_1(K) = X \setminus K$ and thus the set is Chebyshev and by Lemma 6.3.13 we also have $\partial^0 d_K(x)$ is singleton for each $x \in X \setminus K$. Thus the result follows as a simple corollary of Theorem 6.4.6.

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